

A RELATIONSHIP BETWEEN CHARACTERISTIC VALUES AND VECTORS

E. T. BEASLEY, JR. AND P. M. GIBSON

ABSTRACT. It is shown that for all nonzero n -component column vectors α and β over a field F there exists a set Γ of n -square matrices over F of cardinality $n^2 - 2n + 2$ such that, for each n -square matrix A over F , $A\alpha = \alpha$ or $A^T\beta = \beta$ if and only if 1 is a characteristic value of PA for every $P \in \Gamma$.

Let F be a field with unit 1, and A an n -square matrix over F . If all row sums of A or all row sums of A^T (the transpose of A) are 1, then A is a *stochastic* matrix. It is obvious that if A is stochastic then 1 is a characteristic value of PA for every n -square permutation matrix P . R. A. Brualdi and H. W. Wielandt [1] proved the converse. In this paper, it is shown as a corollary to a more general result that the $n!$ permutation matrices can be replaced by a set of n -square matrices of cardinality $n^2 - 2n + 2$.

Let F^n be the set of all $n \times 1$ column vectors over F , and let $\alpha, \beta \in F^n - \{0\}$. A set Γ of n -square matrices over F is called an (α, β) -set provided that, for each n -square matrix A over F , $A\alpha = \alpha$ or $A^T\beta = \beta$ if and only if 1 is a characteristic value of PA for every $P \in \Gamma$. An (α, β) -set Γ is *minimal* if no proper subset of Γ is an (α, β) -set. If $\alpha^T\beta = 0$, then α and β are *orthogonal*. Let ε_k be the vector in F^n with all components zero, except component k , which is equal to 1. Denote the vector in F^n with all components equal to 1 by ε . The characterization of stochastic matrices by Brualdi and Wielandt implies that the set of all n -square permutation matrices is an $(\varepsilon, \varepsilon)$ -set of cardinality $n!$. We shall show that minimal (α, β) -sets of cardinality $n^2 - 2n + 2$ can be easily constructed for all $\alpha, \beta \in F^n - \{0\}$.

We first exhibit minimal (α, β) -sets for two particular choices of α and β . One of these is $\alpha = \beta = \varepsilon_1$, and the other is $\alpha = \varepsilon_1$ and $\beta = \varepsilon_n$.

LEMMA 1. For each positive integer n , the set

$$\Phi_n = \{\varepsilon_1\varepsilon_1^T\} \cup \{\varepsilon_1\varepsilon_1^T + \varepsilon_i\varepsilon_j^T \mid i, j = 2, 3, \dots, n\}$$

of n -square matrices over F is a minimal $(\varepsilon_1, \varepsilon_1)$ -set.

Presented to the Society November 17, 1973; received by the editors March 28, 1973.

AMS (MOS) subject classifications (1970). Primary 15A18; Secondary 15A51.

© American Mathematical Society 1974

PROOF. Let $A=(a_{ij})$ be an n -square matrix over F . Since $\varepsilon_1^T \varepsilon_1=1$ and $\varepsilon_j^T \varepsilon_1=0$ for $j=2, 3, \dots, n$,

$$A\varepsilon_1 = \varepsilon_1 \Rightarrow PA\varepsilon_1 = \varepsilon_1 \quad \forall P \in \Phi_n,$$

$$A^T \varepsilon_1 = \varepsilon_1 \Rightarrow (PA)^T \varepsilon_1 = \varepsilon_1 \quad \forall P \in \Phi_n.$$

Hence, if $A\varepsilon_1=\varepsilon_1$ or $A^T \varepsilon_1=\varepsilon_1$, then 1 is a characteristic value of PA for every $P \in \Phi_n$. Now suppose that 1 is a characteristic value of PA for every $P \in \Phi_n$. Since 1 is a characteristic value of $\varepsilon_1 \varepsilon_1^T A$, $a_{11}=1$. Since 1 is a characteristic value of $(\varepsilon_1 \varepsilon_1^T + \varepsilon_j \varepsilon_j^T)A$,

$$\det((\varepsilon_1 \varepsilon_1^T + \varepsilon_j \varepsilon_j^T)A - I) = 0, \quad i, j = 2, 3, \dots, n.$$

This implies that

$$(1) \quad \det \begin{bmatrix} a_{11} - 1 & a_{1j} \\ a_{i1} & a_{ij} - 1 \end{bmatrix} = 0, \quad i, j = 2, 3, \dots, n.$$

Therefore, since $a_{11}=1$, $a_{i1}a_{1j}=0$, $i, j=2, 3, \dots, n$. Since $a_{11}=1$, this implies that $A\varepsilon_1=\varepsilon_1$ or $A^T \varepsilon_1=\varepsilon_1$. Hence, Φ_n is an $(\varepsilon_1, \varepsilon_1)$ -set. We now show that Φ_n is minimal. If

$$A = \varepsilon \varepsilon_1^T + \varepsilon_1 \varepsilon^T - 2\varepsilon_1 \varepsilon_1^T,$$

then $A\varepsilon_1=A^T \varepsilon_1 \neq \varepsilon_1$, and equation (1) holds. This implies that $\Phi_n - \{\varepsilon_1 \varepsilon_1^T\}$ is not an $(\varepsilon_1, \varepsilon_1)$ -set. If $k, m \in \{2, 3, \dots, n\}$ and

$$A = \varepsilon_1 \varepsilon_1^T + \varepsilon_1 \varepsilon_k^T + \varepsilon_m \varepsilon_1^T,$$

then $A\varepsilon_1 \neq \varepsilon_1 \neq A^T \varepsilon_1$, $a_{11}=1$, and equation (1) holds except for $i=m$ and $j=k$. This implies that

$$\Phi_n - \{\varepsilon_1 \varepsilon_1^T + \varepsilon_k \varepsilon_m^T\}, \quad k, m = 2, 3, \dots, n,$$

is not an $(\varepsilon_1, \varepsilon_1)$ -set. Therefore, Φ_n is a minimal $(\varepsilon_1, \varepsilon_1)$ -set.

LEMMA 2. Let $E_{1n}=\varepsilon_1 \varepsilon_1^T + \varepsilon_n \varepsilon_n^T$. For each integer $n>1$, the set

$$\Psi_n = \{E_{1n}, E_{1n} + \varepsilon_1 \varepsilon_n^T\} \\ \cup \{E_{1n} + \varepsilon_1 \varepsilon_j^T, E_{1n} + \varepsilon_i \varepsilon_n^T, E_{1n} + \varepsilon_i \varepsilon_j^T + \varepsilon_i \varepsilon_n^T \mid i, j = 2, 3, \dots, n-1\}$$

of n -square matrices over F is a minimal $(\varepsilon_1, \varepsilon_n)$ -set.

PROOF. Let $A=(a_{ij})$ be an n -square matrix over F . Then

$$A\varepsilon_1 = \varepsilon_1 \Rightarrow E_{1n}A\varepsilon_1 = \varepsilon_1, \varepsilon_i \varepsilon_{j+1}^T A\varepsilon_1 = 0, \quad i, j = 1, 2, \dots, n-1,$$

$$A^T \varepsilon_n = \varepsilon_n \Rightarrow (E_{1n}A)^T \varepsilon_n = \varepsilon_n, (\varepsilon_i \varepsilon_{j+1}^T A)^T \varepsilon_n = 0, \quad i, j = 1, 2, \dots, n-1.$$

Hence, if $A\varepsilon_1 = \varepsilon_1$ or $A^T\varepsilon_n = \varepsilon_n$, then 1 is a characteristic value of PA for every $P \in \Psi_n$. Suppose that 1 is a characteristic value of PA for every $P \in \Psi_n$. Since 1 is a characteristic value of $E_{1n}A$ and $(E_{1n} + \varepsilon_1\varepsilon_n^T)A$,

$$(2) \quad \det \begin{bmatrix} a_{11} - 1 & a_{1n} \\ a_{n1} & a_{nn} - 1 \end{bmatrix} = 0,$$

$$(3) \quad \det \begin{bmatrix} a_{11} + a_{n1} - 1 & a_{1n} + a_{nn} \\ a_{n1} & a_{nn} - 1 \end{bmatrix} = 0.$$

Equations (2) and (3) imply that $a_{n1} = 0$. Then equation (2) implies that $a_{11} = 1$ or $a_{nn} = 1$. Hence, if $n = 2$, then $A\varepsilon_1 = \varepsilon_1$ or $A^T\varepsilon_n = \varepsilon_n$. Suppose that $n > 2$. We complete the proof that Φ_n is an $(\varepsilon_1, \varepsilon_n)$ -set by considering three cases.

CASE 1. $a_{11} = 1, a_{nn} \neq 1$. Since 1 is a characteristic value of $(E_{1n} + \varepsilon_1\varepsilon_i^T)A$,

$$(4) \quad \det \begin{bmatrix} a_{11} + a_{i1} - 1 & a_{1n} + a_{in} \\ a_{n1} & a_{nn} - 1 \end{bmatrix} = 0, \quad i = 2, 3, \dots, n - 1.$$

Since $a_{n1} = 0, a_{11} = 1$, and $a_{nn} \neq 1$, equation (4) implies that $a_{i1} = 0$ for $i = 2, 3, \dots, n - 1$. Therefore, $A\varepsilon_1 = \varepsilon_1$.

CASE 2. $a_{11} \neq 1, a_{nn} = 1$. Since 1 is a characteristic value of $(E_{1n} + \varepsilon_j\varepsilon_n^T)A$,

$$(5) \quad \det \begin{bmatrix} a_{11} - 1 & a_{1j} & a_{1n} \\ a_{n1} & a_{nj} - 1 & a_{nn} \\ a_{n1} & a_{nj} & a_{nn} - 1 \end{bmatrix} = 0, \quad j = 2, 3, \dots, n - 1.$$

Since $a_{n1} = 0, a_{11} \neq 1$, and $a_{nn} = 1$, equation (5) implies that $a_{nj} = 0$ for $j = 2, 3, \dots, n - 1$. Hence, $A^T\varepsilon_n = \varepsilon_n$.

CASE 3. $a_{11} = 1 = a_{nn}$. Since 1 is a characteristic value of

$$(E_{1n} + \varepsilon_1\varepsilon_i^T + \varepsilon_j\varepsilon_n^T)A,$$

$$(6) \quad \det \begin{bmatrix} a_{11} + a_{i1} - 1 & a_{1j} + a_{ij} & a_{1n} + a_{in} \\ a_{n1} & a_{nj} - 1 & a_{nn} \\ a_{n1} & a_{nj} & a_{nn} - 1 \end{bmatrix} = 0$$

for $i, j = 2, 3, \dots, n - 1$. Therefore, since $a_{n1} = 0$ and $a_{11} = 1 = a_{nn}$,

$$a_{i1}a_{nj} = 0, \quad i, j = 2, 3, \dots, n - 1,$$

and we see that $A\varepsilon_1 = \varepsilon_1$ or $A^T\varepsilon_n = \varepsilon_n$.

We now show that Ψ_n is minimal. If

$$A = \sum_{k=2}^n \varepsilon_k \varepsilon_1^T + \sum_{m=2}^{n-1} \varepsilon_1 \varepsilon_m^T - \sum_{k,m=2}^{n-1} \varepsilon_k \varepsilon_m^T,$$

then it is not difficult to show that $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (3) through (6) hold. This implies that $\Psi_n - \{E_{1n}\}$ is not an $(\varepsilon_1, \varepsilon_n)$ -set. If $A = \varepsilon_1 \varepsilon_n^T + \varepsilon_n \varepsilon_1^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2), (4), (5), and (6) hold. This implies that $\Psi_n - \{E_{1n} + \varepsilon_1 \varepsilon_n^T\}$ is not an $(\varepsilon_1, \varepsilon_n)$ -set. If $m \in \{2, 3, \dots, n-1\}$ and $A = \varepsilon_1 \varepsilon_1^T + \varepsilon_m \varepsilon_1^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2) through (6) hold except when $i=m$ in equation (4). This implies that

$$\Psi_n - \{E_{1n} + \varepsilon_1 \varepsilon_m^T\}, \quad m = 2, 3, \dots, n-1,$$

is not an $(\varepsilon_1, \varepsilon_n)$ -set. If $k \in \{2, 3, \dots, n-1\}$ and $A = \varepsilon_n \varepsilon_k^T + \varepsilon_n \varepsilon_n^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2) through (6) hold except when $j=k$ in equation (5). This implies that

$$\Psi_n - \{E_{1n} + \varepsilon_k \varepsilon_n^T\}, \quad k = 2, 3, \dots, n-1,$$

is not an $(\varepsilon_1, \varepsilon_n)$ -set. If $k, m \in \{2, 3, \dots, n-1\}$ and $A = E_{1n} + \varepsilon_m \varepsilon_1^T + \varepsilon_n \varepsilon_k^T$, then $A\varepsilon_1 \neq \varepsilon_1$ and $A^T \varepsilon_n \neq \varepsilon_n$, while equations (2) through (6) hold except when $i=m$ and $j=k$ in equation (6). This implies that

$$\Psi_n - \{E_{1n} + \varepsilon_1 \varepsilon_m^T + \varepsilon_k \varepsilon_n^T\}, \quad k, m = 2, 3, \dots, n-1,$$

is not an $(\varepsilon_1, \varepsilon_n)$ -set. Hence, Ψ_n is a minimal $(\varepsilon_1, \varepsilon_n)$ -set.

LEMMA 3. Let $\alpha, \beta, \gamma, \delta \in F^n - \{0\}$, and let P and Q be nonsingular matrices over F such that

$$(7) \quad P\gamma = \alpha, \quad P^T\beta = c\delta, \quad Q\alpha = \gamma, \quad Q^Tc\delta = \beta,$$

where $c \in F - \{0\}$. If Γ and Γ' are sets of n -square matrices over F such that

$$(8) \quad \Gamma' = \{PBQ \mid B \in \Gamma\},$$

then Γ' is an (α, β) -set if and only if Γ is a (γ, δ) -set.

PROOF. Suppose that Γ is a (γ, δ) -set, and let A be an n -square matrix over F . Assume that

$$A\alpha = \alpha \quad \text{or} \quad A^T\beta = \beta.$$

Since (7) holds, this implies that $AP\gamma = Q^{-1}\delta$ or $A^TQ^T\delta = (P^T)^{-1}\delta$. Therefore,

$$(QAP)\gamma = \gamma \quad \text{or} \quad (QAP)^T\delta = \delta.$$

Hence, since Γ is a (γ, δ) -set, 1 is a characteristic value of $B(QAP)$ for every $B \in \Gamma$. If $G = PBQ$, then $B(QAP)$ and GA are similar matrices. Therefore, from (8) we see that 1 is a characteristic value of GA for every $G \in \Gamma'$. Reversing this sequence of steps, we see that $A\alpha = \alpha$ or $A^T\beta = \beta$ if 1 is a characteristic value of GA for every $G \in \Gamma'$. Therefore, Γ' is an (α, β) -set. A similar argument shows that Γ is a (γ, δ) -set if Γ' is an (α, β) -set.

It is convenient to express our principal results on minimal (α, β) -sets in two theorems. One of these applies when $\alpha^T\beta \neq 0$, and the other when $\alpha^T\beta = 0$.

THEOREM 1. *Let $\alpha, \beta \in F^n$ with $\alpha^T\beta \neq 0$. If $\{\gamma_i, \delta_i | i = 1, 2, \dots, n-1\}$ is a subset of F^n such that $\{\alpha, \gamma_1, \gamma_2, \dots, \gamma_{n-1}\}$ and $\{\beta, \delta_1, \delta_2, \dots, \delta_{n-1}\}$ are independent while α and β are orthogonal to δ_i and γ_i , respectively, for $i = 1, 2, \dots, n-1$, then*

$$\Gamma = \{(\alpha^T\beta)^{-1}\alpha\beta^T\} \cup \{(\alpha^T\beta)^{-1}\alpha\beta^T + \gamma_i\delta_j^T \mid i, j = 1, 2, \dots, n-1\}$$

is a minimal (α, β) -set.

PROOF. Let P be the n -square matrix with columns $\alpha, \gamma_1, \gamma_2, \dots, \gamma_{n-1}$, and let Q be the n -square matrix with columns $\beta', \delta_1, \delta_2, \dots, \delta_{n-1}$, where $\beta' = (\alpha^T\beta)^{-1}\beta$. Then P and Q are nonsingular with

$$P\varepsilon_1 = \alpha, \quad P^T\beta = (\alpha^T\beta)\varepsilon_1, \quad Q^T\alpha = \varepsilon_1, \quad Q(\alpha^T\beta)\varepsilon_1 = \beta.$$

Since $\Gamma = \{PBQ^T \mid B \in \Phi_n\}$, it follows from Lemmas 1 and 3 that Γ is a minimal (α, β) -set.

THEOREM 2. *Let $\alpha, \beta \in F^n - \{0\}$ with $\alpha^T\beta = 0$. Suppose that*

$$\{\gamma_i, \delta_i \mid i = 1, 2, \dots, n-1\}$$

is a subset of F^n such that $\{\alpha, \gamma_1, \gamma_2, \dots, \gamma_{n-1}\}$ and $\{\beta, \delta_1, \delta_2, \dots, \delta_{n-1}\}$ are independent, $\alpha^T\delta_1 = 1 = \beta^T\gamma_{n-1}$, and α and β are orthogonal to δ_{i+1} and γ_i , respectively, for $i = 1, 2, \dots, n-2$. If $H = \alpha\delta_1^T + \gamma_{n-1}\beta^T$, then

$$\Gamma = \{H, H + \alpha\beta^T\} \\ \cup \{H + \alpha\delta_{j+1}^T, H + \gamma_i\beta^T, H + \alpha\delta_{j+1}^T + \gamma_i\beta^T \mid i, j = 1, 2, \dots, n-2\}$$

is a minimal (α, β) -set.

PROOF. Let P be the n -square matrix with columns $\alpha, \gamma_1, \gamma_2, \dots, \gamma_{n-1}$, and let Q be the n -square matrix with columns $\delta_1, \delta_2, \dots, \delta_{n-1}, \beta$. Then P and Q are nonsingular with

$$P\varepsilon_1 = \alpha, \quad P^T\beta = \varepsilon_n, \quad Q^T\alpha = \varepsilon_1, \quad Q\varepsilon_n = \beta.$$

Since $\Gamma = \{PBQ^T \mid B \in \Psi_n\}$, it follows from Lemmas 2 and 3 that Γ is a minimal (α, β) -set.

Let $\alpha, \beta \in F^n - \{0\}$. Clearly a subset $\{\gamma_i, \delta_i \mid i=1, 2, \dots, n-1\}$ of F^n exists satisfying the conditions of Theorems 1 or 2 according to whether $\alpha^T \beta \neq 0$ or $\alpha^T \beta = 0$. Hence we have the following.

COROLLARY 1. *If $\alpha, \beta \in F^n - \{0\}$, there exists a minimal (α, β) -set of cardinality $n^2 - 2n + 2$.*

If we let $\alpha = \beta = \varepsilon$ in this corollary we obtain the following.

COROLLARY 2. *There exists a set Γ of n -square matrices over F of cardinality $n^2 - 2n + 2$ such that, for each n -square matrix A over F , A is stochastic if and only if 1 is a characteristic value of PA for every $P \in \Gamma$.*

Consideration of (α, β) -sets was motivated by Brualdi and Wielandt's remark [1] on the difficulty of finding sets of fewer than $n!$ permutation matrices which could be used in their characterization of the n -square stochastic matrices. We have no general results on this problem, but we can show that every set of five 3-square permutation matrices is a minimal $(\varepsilon, \varepsilon)$ -set.

Let Γ be the set of all 3-square permutation matrices over F , excluding the identity matrix. Let $A = (a_{ij})$ be a 3-square matrix over F . Suppose that 1 is a characteristic value of PA for every $P \in \Gamma$. Then

$$\det(I - PA) = 0 \quad \forall P \in \Gamma.$$

If we multiply each matrix $I - PA$ on the left by P^T , we obtain

$$(9) \quad \det(P - A) = 0 \quad \forall P \in \Gamma.$$

Let

$$\Delta = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

If we add the second and third row of $P - A$ to the first row of $P - A$, and then add the second and third column to the first column we see from (9) that

$$(10) \quad \det \begin{bmatrix} s & c_2 & c_3 \\ r_2 & p_{22} - a_{22} & p_{23} - a_{23} \\ r_3 & p_{32} - a_{32} & p_{33} - a_{33} \end{bmatrix} = 0$$

for every matrix $\begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix}$ in Δ , where

$$s = 3 - \sum_{i,j=1}^3 a_{ij}, \quad c_k = 1 - \sum_{i=1}^3 a_{ik}, \quad r_k = 1 - \sum_{j=1}^3 a_{kj}, \quad k = 2, 3.$$

Suppose that $s \neq 0$. Then (10) must hold for $c_2 = c_3 = 0$, since any multiple of the first column can be added to any other column without changing the determinant. If $s \neq 0$ and $c_2 = c_3 = 0$ in (10), we see that

$$(11) \quad \begin{aligned} a_{22}a_{33} - (a_{23} - 1)(a_{32} - 1) &= 0, \\ a_{22}a_{33} - (a_{23} - 1)a_{32} &= 0, \\ a_{22}a_{33} - a_{23}(a_{32} - 1) &= 0, \\ (a_{22} - 1)a_{33} - a_{23}a_{32} &= 0, \\ a_{22}(a_{33} - 1) - a_{23}a_{32} &= 0. \end{aligned}$$

It is not difficult to show that this system of equations has no solution. Hence, $s = 0$.

Since $s = 0$, from (10) we obtain a homogeneous system of five linear equations in the four unknowns $c_2r_2, c_2r_3, c_3r_2, c_3r_3$. Since the coefficient matrix for this system has rank four, we see that

$$r_2 = r_3 = 0 \quad \text{or} \quad c_2 = c_3 = 0.$$

Therefore, since $s = 0$, $A\varepsilon = \varepsilon$ or $A^T\varepsilon = \varepsilon$. Therefore, Γ is an $(\varepsilon, \varepsilon)$ -set. If any equation is removed from system (11), then the remaining system has a solution. Hence Γ is a minimal $(\varepsilon, \varepsilon)$ -set. It now follows from Lemma 3 that every set of five 3-square permutation matrices is a minimal $(\varepsilon, \varepsilon)$ -set.

We have determined by machine calculation that the 12 matrices

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ &\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\ &\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

over $GF(2)$ form a minimal $(\varepsilon, \varepsilon)$ -set. There also exist minimal $(\varepsilon, \varepsilon)$ -sets of 4-square permutation matrices over $GF(2)$ of cardinality 13, but there exist none of cardinality less than 12.

REFERENCE

1. R. A. Brualdi and H. W. Wielandt, *A spectral characterization of stochastic matrices*, *Linear Algebra and Appl.* **1** (1968), no. 1, 65–71. MR **36** #6435.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, HUNTSVILLE, ALABAMA
35807 (Current address of P. M. Gibson)

Current address (E. T. Beasley): Department of Computer Science, Ohio State
University, Columbus, Ohio 43210