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ON KAEHLER MANIFOLDS SATISFYING THE AXIOM OF ANTIHOLOMORPHIC 2-SPHERES

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ABSTRACT. A Kaehler manifold with the axiom of antiholomorphic 2-spheres is a complex space form.

1. **Introduction.** Let M be a Kaehler manifold with complex structure J and Riemann metric g.

By a *plane section* we mean a 2-dimensional linear subspace of a tangent space. A plane section π is called *holomorphic* (resp. *antiholomorphic*) if $J\pi=\pi$ (resp. $J\pi$ is perpendicular to π). The sectional curvature for a holomorphic (resp. antiholomorphic) plane section is called holomorphic (resp. antiholomorphic) sectional curvature.

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. It is well known that a complex space form has constant antiholomorphic sectional curvature.

Conversely, in their recent paper [**1**], B. Y. Chen and K. Ogiue proved that a Kaehler manifold with dimension $\geqq 3$ and constant antiholomorphic sectional curvature is a complex space form.

A Kaehler manifold M is said to satisfy the *axiom of holomorphic planes* (resp. *axiom of antihomomorphic planes*) if, for each $x\in M$ and each holomorphic (resp. antiholomorphic) plane $\pi\subset T_x(M)$, there exists a 2-dimensional totally geodesic submanifold N such that $x\in N$ and $T_x(N)=\pi$. I. Mogi and K. Yano [**4**] proved that a Kaehler manifold with the axiom of holomorphic planes is a complex space form.

Recently, B. Y. Chen and K. Ogiue [**1**] proved that a Kaehler manifold with dimension $\geqq 3$ and the axiom of antiholomorphic planes is a complex space form.

A Riemannian manifold M of (real) dimension $\geqq 3$ is said to satisfy the *axiom of* 2-*spheres* if, for each $x\in M$ and each plane $\pi\subset T_x(M)$, there exists a 2-dimensional umbilical submanifold N with parallel mean curvature vector field such that $x\in N$ and $T_x(N)=\pi$. D. Leung and K. Nomizu [**3**] proved that a manifold with this property has constant sectional curvature.

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Recently, in his paper [2], S. Goldberg introduced the *axiom of holomorphic 2-spheres*; a Hermitian manifold M is said to satisfy the axiom of holomorphic 2-sphere if, for each $x \in M$ and each holomorphic plane $\pi \subset T_x(M)$, there exists a 2-dimensional umbilical submanifold N with parallel mean curvature vector field such that $x \in N$ and $T_x(N) = \pi$. He proved that a Kaehler manifold satisfying the axiom of holomorphic 2-spheres has constant holomorphic sectional curvature.

A Kaehler manifold is said to satisfy the *axiom of antiholomorphic 2-spheres* if, for each $x \in M$ and each antiholomorphic plane $\pi \subset T_x(M)$, there exists a 2-dimensional umbilical submanifold N with parallel mean curvature vector field such that $x \in N$ and $T_x(N) = \pi$.

We shall prove the following theorem in this paper.

THEOREM. *Let M be a Kaehler manifold. If M satisfies the axiom of antiholomorphic 2-spheres and if* $\dim M \geqq 3$, *then M is a complex space form.*

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2. **Preliminaries.** Let M be a Kaehler manifold with complex structure J and Riemann metric g. We denote by R the curvature tensor field of M. Then we have

$$R(JX, JY) = R(X, Y), \tag{2.1}$$

$$R(X, Y)JZ = JR(X, Y)Z. \tag{2.2}$$

Let $K(X, Y)$ be the sectional curvature of M determined by orthonormal vectors X and Y. Then we have

$$K(JX, JY) = K(X, Y), \tag{2.3}$$

$$K(X, JY) = K(JX, Y). \tag{2.4}$$

The following is easily seen.

(2.5) *Orthonormal vectors X and Y span an antiholomorphic section if and only if X, Y and JX are orthonormal.*

Let N be a submanifold of M and let $\tilde{\nabla}$ and ∇ be the covariant differentiations on M and N respectively. Then the second fundamental form σ of the immersion is defined by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

where X and Y are vector fields tangent to N. σ is a normal bundle valued symmetric 2-form on N. For a vector field ξ normal to N we write $\tilde{\nabla}_X\xi = -A_\xi X + D_X\xi$, where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X\xi$. The tensor fields σ and A_ξ are related by

$$g(\sigma(X, Y), \xi) = g(A_\xi X, Y). \tag{2.6}$$

Since $R(X, Y)\xi = \tilde{\nabla}_X\tilde{\nabla}_Y\xi - \tilde{\nabla}_Y\tilde{\nabla}_X\xi - \tilde{\nabla}_{[X,Y]}\xi$, we can obtain

$$\begin{aligned} R(X, Y)\xi &= (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y \\ &\quad + A_{D_X\xi}Y - A_{D_Y\xi}X \quad \text{(modulo normal component).} \end{aligned} \tag{2.7}$$

The *mean curvature normal* H of N in M is defined by the relation

$$\operatorname{trace} A_\xi = 2g(\xi, H), \tag{2.8}$$

for all ξ normal to N. It is called *parallel* (in the normal bundle) if $DH=0$. The surface N is *umbilical* in M if $\sigma(X, Y) = g(X, Y)H$, i.e., if

$$A_\xi = g(\xi, H)I = \tfrac{1}{2}(\operatorname{trace} A_\xi)I, \tag{2.9}$$

where I is the identity transformation.

An umbilical submanifold is totally geodesic if H vanishes.

3. **Proof of theorem.** Let x be an arbitrary point of M and let X and Y be arbitrary orthonormal vectors in $T_x(M)$ which span an anti-holomorphic section π. Then, there is an umbilical submanifold N with parallel mean curvature normal H such that $x \in N$ and $T_x(N) = \pi$. Let U be a normal neighborhood of x in N and for each $y \in U$ let ξ_y be the normal to N at y parallel (with respect to D) to JX along the geodesic in U from x to y. Along each such geodesic, $g(\xi, H)$ is a constant c, since ξ and H are parallel. Therefore (2.9) implies that $A_\xi = cI$ at every point of U. Thus

$$\begin{aligned} \nabla_X A_\xi &= \nabla_Y A_\xi = 0, \\ D_X\xi &= D_Y\xi = 0 \quad \text{at } x. \end{aligned}$$

From them, together with (2.7), we obtain $R(X, Y)JX = 0$ (modulo normal component). In particular we have $g(R(X, Y)JX, X) = 0$. Now our theorem follows from the following.

LEMMA ([1], [5]). *If* $g(R(X, Y)JX, X) = 0$ *for every orthonormal* $X, Y, JX \in T_x(M)$ *and for every point* x *of* M, *then* M *is a complex space form, provided that* dim $M \geqq 3$.

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