

ON PANOV'S THEOREM

PETER S. LANDWEBER<sup>1</sup>

ABSTRACT. We give a simple proof of Panov's theorem, which determines the elements of  $H_*(MU)$  mapped into  $\pi_*(MU)$  by all operations  $s_\omega$  for  $\omega > 0$ .

The purpose of this note is to present a simple proof of the main theorem of N. V. Panov's paper *Characteristic numbers in U-theory* [6]. In addition, Panov goes on to obtain complete results concerning the Chern numbers of  $(U, fr)$ -manifolds; see §4 of [6].

Let  $MU_*(X)$  denote the complex bordism of a space or spectrum  $X$ . Then there are stable operations

$$s_\omega : MU_*(X) \rightarrow MU_*(X)$$

for each partition  $\omega$ ; if  $\omega$  is a partition of  $n$  ( $|\omega|=n$ ) then  $s_\omega$  lowers degrees by  $2n$  ([4], [5]). We shall always assume that  $|\omega| > 0$  when dealing with the operations  $s_\omega$  (for  $\omega=0$ ,  $s_\omega$  is the identity, which is of no interest here). An element  $a \in MU_*(X)$  is called primitive if  $s_\omega(a)=0$  for all  $\omega$ .

We may regard  $\pi_*(MU)=MU_*$  as a submodule of  $H_*(MU)$  by means of the Hurewicz homomorphism. If  $H$  denotes the integral Eilenberg-Mac Lane spectrum and  $S$  the sphere spectrum, then

$$H_*(MU)/MU_* \approx MU_*(H/S).$$

Let  $N_1 = \{a \in H_*(MU); s_\omega a \in MU_* \text{ for all } \omega\}$ ; then  $N_1/MU_*$  can be identified with the primitive elements in  $MU_*(H/S)$ . Panov gives a determination of  $N_1$ .

THEOREM (PANOV). *Let  $n > 0$ . Then  $(N_1)_{2n}/MU_{2n}$  is a cyclic group. Moreover,*

- (a) *for  $n$  odd, it has order 2 with generator  $CP(1)^n/2$ ;*
- (b) *for  $n=2$ , it has order 12 with generators  $CP(1)^2/4$  and  $CP(2)/3$ ;*
- (c) *for  $n$  even and  $n > 2$ , it has  $p$ -torsion iff  $n \equiv 0 \pmod{p-1}$ . For such  $p$  write  $k=n/(p-1)=p^m l$ ,  $(p, l)=1$ . If  $p$  is odd, the  $p$ -component has order  $p^{m+1}$  and generator  $CP(p-1)^k/p^{m+1}$ . If  $p=2$ , the 2-primary part has order*

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$2^{m+2}$  and generator

$$CP(1)/2^{m+2} + H_3CP(1)^{n-3}/2$$

where  $H_3$  is a suitable generator of  $MU_6$  (e.g.,  $H_3=2CP(3)+H_{2,2}$ ).

REMARKS 1. In fact Buhštaber ([1], [2]) introduces a filtration

$$MU_* = N_0 \subset N_1 \subset N_2 \subset \dots \subset H_*(MU)$$

where  $N_{i+1}/N_i$  is the module of primitive elements in  $H_*(MU)/N_i$ . This explains the notation.

2. Let  $s_\omega$  denote characteristic numbers in  $K$ -theory, in particular when  $\omega$  is empty  $s_\omega=Td$ , the Todd genus. The following is Panov's Proposition 6.

PROPOSITION.  $N_1$  consists of those elements in  $H_*(MU)$  for which  $s_\omega$  is integral for every  $\omega > 0$ .

Panov overlooked the following simple proof. First, if  $a \in N_1$  and  $\omega > 0$ , then  $s_\omega(a)=Td s_\omega(a) \in Z$  as desired. Conversely, let  $a \in H_*(MU)$  have all  $s_\omega(a)$  integral for  $\omega > 0$ . To show  $a \in N_1$ , fix  $\omega_0 > 0$ ; we will show  $s_{\omega_0}(a) \in MU_*$ . In view of the Hattori-Stong theorem, it suffices to show  $s_{\omega_1} \circ s_{\omega_0}(a) \in Z$  for all  $\omega_1 \geq 0$ . This is the same as  $Td(s_{\omega_1} \circ s_{\omega_0}(a))$ . Since  $s_{\omega_1} \circ s_{\omega_0}$  is a linear combination of  $s_\omega$ 's with  $\omega > 0$  ([4], [5]), this is clear.

3. One concludes that Panov could just as well use  $K$ -theory characteristic numbers. For example, at no point are the operations  $s_\omega$  composed.

4. Panov does not pick his generator  $H_3$  properly, since  $s_3(H_{1,3})$  is 0 and not  $-4$ . A suitable choice is  $H_3=2CP(3)+H_{2,2}$ , since  $s_3(H_3)=2$  and  $H_3$  agrees with  $H_{2,2} \pmod 2$  (see Proposition 3 and the preceding paragraph in [6]).

5. The evident  $BP$  analogue of Panov's theorem is valid. One simply replaces  $CP(p-1)$  with  $v_1$ , and  $H_3$  with  $v_2$  in the interesting case  $p=2$  and  $n$  even,  $n > 2$ .

We now outline our proof of Panov's theorem.

(a) Show  $(N_1)_{2n}/MU_{2n}$  is cyclic and has  $p$ -torsion iff  $n \equiv 0 \pmod{p-1}$ .

(b) For such  $p$ , write  $n=(p-1)k$  and  $k=p^ml$  with  $(p,l)=1$ . Show that  $CP(p-1)^k/p^{m+1}$  lies in  $(N_1)_{2n}$ , and unless  $p=2$  and  $n$  is even and  $n > 2$  it generates the  $p$ -component.

(c) If  $n$  is even and  $n > 2$ , show that  $P=CP(1)^n/2^{m+2}+H_3CP(1)^{n-3}/2$  lies in  $(N_1)_{2n}$  and generates the 2-component of  $(N_1)_{2n}/MU_{2n}$ .

By proving (a) first, one avoids the need for Panov's splittings. (b) is included in Proposition 4 of [6]. Our main contribution is in the proof of (c).

At this point it is convenient to follow [6] and choose generators  $\{H_n\}$  for  $MU_*$ ,  $H_n \in MU_{2n}$ , such that  $H_{p-1}=CP(p-1)$  and, if  $k > 0$ ,

$H_{p^{k+1}-1}$  is a nonzero multiple mod  $p$  of Stong's hypersurface ([3], [7])

$$H_{p^k, \dots, p^k} \subset CP(p^k) \times \dots \times CP(p^k)$$

(where  $p^k$  occurs  $p$  times). In particular,  $H_3 \equiv H_{2,2} \pmod 2$ .

For a fixed prime  $p$  and  $a \in N_1$ , we say that  $a$  has type  $\omega_0$  if  $s_{\omega_0}(a) \not\equiv 0 \pmod p$  and  $s_\omega(a) \equiv 0 \pmod p$  if  $\omega > \omega_0$ . Recall that one puts  $\omega > \omega_0$  if  $|\omega| > |\omega_0|$ , or if  $|\omega| = |\omega_0|$  and  $\omega$  involves fewer terms than  $\omega_0$ ; see [3], [7]. For example,  $H_{p-1}$  has type 0;  $H_{p^{k+1}-1}$  has type  $(p^k-1, \dots, p^k-1)$  if  $k > 0$  (where  $p^k-1$  occurs  $p$  times); and  $H_n$  has type  $(n)$  if  $n+1$  is not a power of  $p$ . For the  $H_{p^{k+1}-1}$  this is due to Stong ([3], [7]) if one computes with  $K$ -theory characteristic numbers. The Chern-Dold character makes it possible to carry over the computation to  $MU$ -characteristic numbers; see Proposition 2 of [6].

Notice that one has  $s_\omega CP(1)^n = 0$  if  $\omega \neq (1, \dots, 1)$ . As is customary, write  $s_\omega = s^{k\Delta_1}$  if  $\omega = (1, \dots, 1)$  with  $k$  1's. Then we have

$$s^{k\Delta_1} CP(1)^n = \binom{n}{k} 2^k CP(1)^{n-k},$$

and so it is convenient to notice the following property of binomial coefficients. We write  $\nu_p(r)$  for the highest exponent of  $p$  dividing the integer  $r$ .

LEMMA Let  $n = p^m l$ ,  $(l, p) = 1$ , where  $p$  is a prime. If  $k \leq p^m$  then  $\nu_p \binom{n}{k} = m - \nu_p(k)$ .

PROOF. Simply notice that

$$\binom{n}{k} = \frac{p^{ml}}{k} \prod_{i=1}^{k-1} \frac{p^{ml} - i}{i}$$

and that  $\nu_p(p^{ml} - i) = \nu_p(i)$  in this range.

PROOF OF (a). We show that in  $(N_1)_{2n}/MU_{2n}$  the elements of order  $p$  must be multiples of  $CP(p-1)^k/p$ , where  $k = n/(p-1)$  is assumed to be integral. For let  $a \in (N_1)_{2n}$ ,  $pa \in MU_{2n}$ ,  $a \notin MU_{2n}$ . Write  $a = \lambda CP(p-1)^k + \sum \lambda_I H_I$  with  $\lambda$  and  $\lambda_I$  rational (the  $H_I$  are monomials in the generators of  $MU_*$ ). Since  $pa \in MU_{2n}$ ,  $p\lambda$  and  $p\lambda_I$  are integral. An argument with types (compare [3, §14]) now shows that  $p\lambda_I \equiv 0 \pmod p$ , hence all  $\lambda_I \in \mathbb{Z}$ . Since

$$a = p\lambda(CP(p-1)^k/p) + \sum \lambda_I H_I,$$

$a$  has the desired form.

PROOF OF (b). A computation using the lemma shows easily that, for  $\omega > 0$ ,  $s_\omega(CP(p-1)^k) \equiv 0 \pmod{p^{m+1}}$ , so  $CP(p-1)^k/p^{m+1}$  belongs to  $N_1$ .

For  $p > 2$  one computes that  $CP(p-1)^k/p^{m+1}$  has type  $(p-1)$ . This also holds for  $p=2$  and  $m=0$  (i.e.,  $n$  odd). For  $p=2$  and  $m > 0$  one finds that  $CP(1)^n/2^{m+1}$  has type  $(1, 1)$ ; explicitly,  $s_1(CP(1)^n/2^{m+1}) \equiv CP(1)^{n-1} \pmod 2$ ,  $s_{1,1}(CP(1)^n/2^{m+1}) \equiv CP(1)^{n-2} \pmod 2$ , and  $s_\omega(CP(1)^n/2^{m+1}) \equiv 0 \pmod 2$  if  $\omega > (1, 1)$ . These observations make up Proposition 4 of [6], hence we do not offer more details.

PROOF OF (c). Let  $p=2$ ,  $n$  even and  $n > 2$ . Write  $n=2^m l$  with  $l$  odd, hence  $m > 0$ . Using the  $MU$ -characteristic numbers of  $H_{2,2}$  listed in Proposition 3 of [6], we learn that  $s_1 H_3 \equiv CP(1) \pmod 2$ ,  $s_{1,1} H_3 \equiv CP(1)^2 \pmod 2$  and  $s_\omega H_3 \equiv 0 \pmod 2$  if  $\omega > (1, 1)$ . It follows that

$$s_\omega[CP(1)^n/2^{m+1} + H_3 CP(1)^{n-3}] \equiv 0 \pmod 2$$

for all  $\omega > 0$ , hence  $P = CP(1)^n/2^{m+2} + H_3 CP(1)^{n-3}/2$  belongs to  $N_1$ .

The next task is to determine the type of  $P$ . We establish a little more:

(d)  $s_3(P) \not\equiv 0 \pmod 2$ ; if  $m=1$  then  $P$  has type (3), and if  $m > 1$  then  $P$  has type  $(1, 1, 1, 1)$ .

To show that (d) implies (c), we must show that it is impossible to have  $s_\omega(P) \equiv s_\omega(a) \pmod 2$  for all  $\omega > 0$  with  $a \in MU_{2n}$ . By an argument with types, this can happen only if  $P$  has type  $(1, 1, 1, 1)$ , i.e.,  $m > 1$ , and then  $a$  must be a linear combination of  $CP(1)^{n-6}(H_3)^2$ ,  $CP(1)^{n-3}H_3$ ,  $CP(1)^{n-2}CP(2)$ , and  $CP(1)^n$ . But then we would have  $s_3(a) \equiv 0 \pmod 2$ , which violates (d).

PROOF OF (d) First of all one computes directly that  $s_3(P) = CP(1)^{n-3}$ , hence  $s_3(P) \not\equiv 0 \pmod 2$ . Thus it remains to consider  $s_\omega(P)$  with  $\omega > (3)$ .

If  $\omega \neq (1, \dots, 1)$ , then

$$s_\omega(P) = \frac{1}{2} s_\omega(H_3 CP(1)^{n-3}).$$

Since  $s_\omega(H_3) = 0$  for dimensional reasons,  $s_{\omega'}(H_3) \equiv 0 \pmod 2$  if  $\omega' \neq (1, \dots, 1)$  and  $s_{\omega''}(CP(1)^{n-3}) = 0$  if  $\omega'' \neq (1, \dots, 1)$  it follows easily that  $s_\omega(P) \equiv 0 \pmod 2$ .

So we need only consider  $s^{k\Delta_1}(P)$  for  $k \geq 4$ . One first shows that

$$s^{k\Delta_1}\{CP(1)^n + 2^{m+1}H_3 CP(1)^{n-3}\} \equiv \binom{n}{k} 2^k CP(1)^{n-k} \pmod{2^{m+3}},$$

hence one wants to know when  $v_2\binom{2m l}{k} + k \geq m+3$ . E.g., if  $m=1$  we want to know when  $v_2\binom{2l}{k} + k \geq 4$ ; since  $k \geq 4$  this always holds, hence  $s^{k\Delta_1}(P) \equiv 0 \pmod 2$  for  $k \geq 4$  when  $m=1$ . This proves (d) for  $m=1$ .

Finally let  $m > 1$ . Note that we may assume  $1 \leq k \leq m+2 \leq 2^m$ . Then the lemma implies that

$$v_2\binom{2m l}{k} = m - v_2(k).$$

Hence  $\nu_2\binom{2m-1}{k} + k \geq m + 3$  iff  $k \geq \nu_2(k) + 3$ , and this fails for  $k=4$ , hence  $s_{1,1,1,1}(M) \not\equiv 0 \pmod{2}$ ; but it holds for  $k > 4$ , which proves (d) for  $m > 1$ . Q.E.D.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903