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ON PANOV'S THEOREM

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ABSTRACT. We give a simple proof of Panov's theorem, which determines the elements of $H_*(MU)$ mapped into $\pi_*(MU)$ by all operations s_ω for $\omega>0$.

The purpose of this note is to present a simple proof of the main theorem of N. V. Panov's paper *Characteristic numbers in U-theory* [6]. In addition, Panov goes on to obtain complete results concerning the Chern numbers of (U,fr)-manifolds; see §4 of [6].

Let $MU_*(X)$ denote the complex bordism of a space or spectrum X. Then there are stable operations

$$s_\omega : MU_*(X) \to MU_*(X)$$

for each partition ω; if ω is a partition of n ($|\omega|=n$) then s_ω lowers degrees by $2n$ ([4], [5]). We shall always assume that $|\omega|>0$ when dealing with the operations s_ω (for $\omega=0$, s_ω is the identity, which is of no interest here). An element $a \in MU_*(X)$ is called primitive if $s_\omega(a)=0$ for all ω.

We may regard $\pi_*(MU)=MU_*$ as a submodule of $H_*(MU)$ by means of the Hurewicz homomorphism. If H denotes the integral Eilenberg-Mac Lane spectrum and S the sphere spectrum, then

$$H_*(MU)/MU_* \approx MU_*(H/S).$$

Let $N_1=\{a \in H_*(MU);\, s_\omega a \in MU_* \text{ for all } \omega\}$; then N_1/MU_* can be identified with the primitive elements in $MU_*(H/S)$. Panov gives a determination of N_1.

THEOREM (PANOV). *Let $n>0$. Then $(N_1)_{2n}/MU_{2n}$ is a cyclic group. Moreover,*

(a) *for n odd, it has order 2 with generator $CP(1)^n/2$;*

(b) *for $n=2$, it has order 12 with generators $CP(1)^2/4$ and $CP(2)/3$;*

(c) *for n even and $n>2$, it has p-torsion iff $n\equiv 0 \bmod (p-1)$. For such p write $k=n/(p-1)=p^m l$, $(p,l)=1$. If p is odd, the p-component has order p^{m+1} and generator $CP(p-1)^k/p^{m+1}$. If $p=2$, the 2-primary part has order*

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2^{m+2} *and generator*

$$CP(1)/2^{m+2} + H_3CP(1)^{n-3}/2$$

where H_3 *is a suitable generator of* MU_6 *(e.g.,* $H_3=2CP(3)+H_{2,2}$*).*

REMARKS 1. In fact Buhštaber ([**1**], [**2**]) introduces a filtration

$$MU_* = N_0 \subset N_1 \subset N_2 \subset \cdots \subset H_*(MU)$$

where N_{i+1}/N_i is the module of primitive elements in $H_*(MU)/N_i$. This explains the notation.

2. Let s_ω denote characteristic numbers in K-theory, in particular when ω is empty $s_\omega = Td$, the Todd genus. The following is Panov's Proposition 6.

PROPOSITION. N_1 *consists of those elements in* $H_*(MU)$ *for which* s_ω *is integral for every* $\omega > 0$.

Panov overlooked the following simple proof. First, if $a \in N_1$ and $\omega > 0$, then $s_\omega(a) = Tds_\omega(a) \in Z$ as desired. Conversely, let $a \in H_*(MU)$ have all $s_\omega(a)$ integral for $\omega > 0$. To show $a \in N_1$, fix $\omega_0 > 0$; we will show $s_{\omega_0}(a) \in MU_*$. In view of the Hattori-Stong theorem, it suffices to show $s_{\omega_1} \circ s_{\omega_0}(a) \in Z$ for all $\omega_1 \geqq 0$. This is the same as $Td(s_{\omega_1} \circ s_{\omega_0}(a))$. Since $s_{\omega_1} \circ s_{\omega_0}$ is a linear combination of s_ω's with $\omega > 0$ ([**4**], [**5**]), this is clear.

3. One concludes that Panov could just as well use K-theory characteristic numbers. For example, at no point are the operations s_ω composed.

4. Panov does not pick his generator H_3 properly, since $s_3(H_{1,3})$ is 0 and not -4. A suitable choice is $H_3 = 2CP(3) + H_{2,2}$, since $s_3(H_3) = 2$ and H_3 agrees with $H_{2,2}$ mod 2 (see Proposition 3 and the preceding paragraph in [**6**]).

5. The evident BP analogue of Panov's theorem is valid. One simply replaces $CP(p-1)$ with v_1, and H_3 with v_2 in the interesting case $p=2$ and n even, $n>2$.

We now outline our proof of Panov's theorem.

(a) *Show* $(N_1)_{2n}/MU_{2n}$ *is cyclic and has p-torsion iff* $n \equiv 0 \bmod (p-1)$.

(b) *For such p, write* $n=(p-1)k$ *and* $k=p^m l$ *with* $(p, l)=1$. *Show that* $CP(p-1)^k/p^{m+1}$ *lies in* $(N_1)_{2n}$, *and unless* $p=2$ *and n is even and* $n>2$ *it generates the p-component.*

(c) *If n is even and* $n>2$, *show that* $P=CP(1)^n/2^{m+2}+H_3CP(1)^{n-3}/2$ *lies in* $(N_1)_{2n}$ *and generates the* 2*-component of* $(N_1)_{2n}/MU_{2n}$.

By proving (a) first, one avoids the need for Panov's splittings. (b) is included in Proposition 4 of [**6**]. Our main contribution is in the proof of (c).

At this point it is convenient to follow [**6**] and choose generators $\{H_n\}$ for MU_*, $H_n \in MU_{2n}$, such that $H_{p-1} = CP(p-1)$ and, if $k>0$,

$H_{p^{k+1}-1}$ is a nonzero multiple mod p of Stong's hypersurface ([3], [7])

$$H_{p^k,\cdots,p^k} \subset CP(p^k) \times \cdots \times CP(p^k)$$

(where p^k occurs p times). In particular, $H_3 \equiv H_{2,2}$ mod 2.

For a fixed prime p and $a \in N_1$, we say that a has *type* ω_0 if $s_{\omega_0}(a) \not\equiv 0 \bmod p$ and $s_\omega(a) \equiv 0 \bmod p$ if $\omega > \omega_0$. Recall that one puts $\omega > \omega_0$ if $|\omega| > |\omega_0|$, or if $|\omega| = |\omega_0|$ and ω involves fewer terms than ω_0; see [3], [7]. For example, H_{p-1} has type 0; $H_{p^{k+1}-1}$ has type $(p^k-1, \cdots, p^k-1)$ if $k>0$ (where p^k-1 occurs p times); and H_n has type (n) if $n+1$ is not a power of p. For the $H_{p^{k+1}-1}$ this is due to Stong ([3], [7]) if one computes with K-theory characteristic numbers. The Chern-Dold character makes it possible to carry over the computation to MU-characteristic numbers; see Proposition 2 of [6].

Notice that one has $s_\omega CP(1)^n = 0$ if $\omega \neq (1, \cdots, 1)$. As is customary, write $s_\omega = s^{k\Delta_1}$ if $\omega = (1, \cdots, 1)$ with k 1's. Then we have

$$s^{k\Delta_1} CP(1)^n = \binom{n}{k} 2^k CP(1)^{n-k},$$

and so it is convenient to notice the following property of binomial coefficients. We write $\nu_p(r)$ for the highest exponent of p dividing the integer r.

LEMMA *Let* $n = p^m l$, $(l, p) = 1$, *where* p *is a prime. If* $k \leq p^m$ *then* $\nu_p\binom{n}{k} = m - \nu_p(k)$.

PROOF. Simply notice that

$$\binom{n}{k} = \frac{p^m l}{k} \prod_{i=1}^{k-1} \frac{p^m l - i}{i}$$

and that $\nu_p(p^m l - i) = \nu_p(i)$ in this range.

PROOF OF (a). We show that in $(N_1)_{2n}/MU_{2n}$ the elements of order p must be multiples of $CP(p-1)^k/p$, where $k = n/(p-1)$ is assumed to be integral. For let $a \in (N_1)_{2n}$, $pa \in MU_{2n}$, $a \notin MU_{2n}$. Write $a = \lambda CP(p-1)^k + \sum \lambda_I H_I$ with λ and λ_I rational (the H_I are monomials in the generators of MU_*). Since $pa \in MU_{2n}$, $p\lambda$ and $p\lambda_I$ are integral. An argument with types (compare [3, §14]) now shows that $p\lambda_I \equiv 0 \bmod p$, hence all $\lambda_I \in \mathbf{Z}$. Since

$$a = p\lambda(CP(p-1)^k/p) + \sum \lambda_I H_I,$$

a has the desired form.

PROOF OF (b). A computation using the lemma shows easily that, for $\omega > 0$, $s_\omega(CP(p-1)^k) \equiv 0 \bmod p^{m+1}$, so $CP(p-1)^k/p^{m+1}$ belongs to N_1.

For $p>2$ one computes that $CP(p-1)^k/p^{m+1}$ has type $(p-1)$. This also holds for $p=2$ and $m=0$ (i.e., n odd). For $p=2$ and $m>0$ one finds that $CP(1)^n/2^{m+1}$ has type $(1, 1)$; explicitly, $s_1(CP(1)^n/2^{m+1}) \equiv CP(1)^{n-1} \bmod 2$, $s_{1,1}(CP(1)^n/2^{m+1}) \equiv CP(1)^{n-2} \bmod 2$, and $s_\omega(CP(1)^n/2^{m+1}) \equiv 0 \bmod 2$ if $\omega > (1, 1)$. These observations make up Proposition 4 of **[6]**, hence we do not offer more details.

PROOF OF (c). Let $p=2$, n even and $n>2$. Write $n=2^m l$ with l odd, hence $m>0$. Using the MU-characteristic numbers of $H_{2,2}$ listed in Proposition 3 of **[6]**, we learn that $s_1 H_3 \equiv CP(1) \bmod 2$, $s_{1,1}H_3 \equiv CP(1)^2 \bmod 2$ and $s_\omega H_3 \equiv 0 \bmod 2$ if $\omega > (1, 1)$. It follows that

$$s_\omega[CP(1)^n/2^{m+1} + H_3 CP(1)^{n-3}] \equiv 0 \bmod 2$$

for all $\omega > 0$, hence $P = CP(1)^n/2^{m+2} + H_3 CP(1)^{n-3}/2$ belongs to N_1.

The next task is to determine the type of P. We establish a little more:

(d) $s_3(P) \not\equiv 0 \bmod 2$; *if* $m=1$ *then* P *has type* (3), *and if* $m>1$ *then* P *has type* (1, 1, 1, 1).

To show that (d) implies (c), we must show that it is impossible to have $s_\omega(P) \equiv s_\omega(a) \bmod 2$ for all $\omega > 0$ with $a \in MU_{2n}$. By an argument with types, this can happen only if P has type (1, 1, 1, 1), i.e., $m>1$, and then a must be a linear combination of $CP(1)^{n-6}(H_3)^2$, $CP(1)^{n-3}H_3$, $CP(1)^{n-2}CP(2)$, and $CP(1)^n$. But then we would have $s_3(a) \equiv 0 \bmod 2$, which violates (d).

PROOF OF (d) First of all one computes directly that $s_3(P) = CP(1)^{n-3}$, hence $s_3(P) \not\equiv 0 \bmod 2$. Thus it remains to consider $s_\omega(P)$ with $\omega > (3)$.

If $\omega \neq (1, \cdots, 1)$, then

$$s_\omega(P) = \tfrac{1}{2} s_\omega(H_3 CP(1)^{n-3}).$$

Since $s_\omega(H_3)=0$ for dimensional reasons, $s_{\omega'}(H_3) \equiv 0 \bmod 2$ if $\omega' \neq (1, \cdots, 1)$ and $s_{\omega''}(CP(1)^{n-3})=0$ if $\omega'' \neq (1, \cdots, 1)$ it follows easily that $s_\omega(P) \equiv 0 \bmod 2$.

So we need only consider $s^{k\Delta_1}(P)$ for $k \geqq 4$. One first shows that

$$s^{k\Delta_1}\{CP(1)^n + 2^{m+1}H_3 CP(1)^{n-3}\} \equiv \binom{n}{k} 2^k CP(1)^{n-k} \bmod 2^{m+3},$$

hence one wants to know when $\nu_2\binom{2ml}{k} + k \geqq m+3$. E.g., if $m=1$ we want to know when $\nu_2\binom{2l}{k} + k \geqq 4$; since $k \geqq 4$ this always holds, hence $s^{k\Delta_1}(P) \equiv 0 \bmod 2$ for $k \geqq 4$ when $m=1$. This proves (d) for $m=1$.

Finally let $m>1$. Note that we may assume $1 \leqq k \leqq m+2 \leqq 2^m$. Then the lemma implies that

$$\nu_2\binom{2ml}{k} = m - \nu_2(k).$$

Hence $\nu_2\binom{2ml}{k}+k\geqq m+3$ iff $k\geqq\nu_2(k)+3$, and this fails for $k=4$, hence $s_{1,1,1,1}(M)\not\equiv 0 \bmod 2$; but it holds for $k>4$, which proves (d) for $m>1$. Q.E.D.

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