ON PANOV'S THEOREM

PETER S. LANDWEBER

Abstract. We give a simple proof of Panov's theorem, which determines the elements of $H_\ast(MU)$ mapped into $\pi_\ast(MU)$ by all operations $s_\omega$ for $\omega > 0$.

The purpose of this note is to present a simple proof of the main theorem of N. V. Panov's paper Characteristic numbers in U-theory [6]. In addition, Panov goes on to obtain complete results concerning the Chern numbers of $(U,fr)$-manifolds; see §4 of [6].

Let $MU_\ast(X)$ denote the complex bordism of a space or spectrum $X$. Then there are stable operations

$$s_\omega: MU_\ast(X) \to MU_\ast(X)$$

for each partition $\omega$; if $\omega$ is a partition of $n$ ($|\omega|=n$) then $s_\omega$ lowers degrees by $2n$ ([4], [5]). We shall always assume that $|\omega|>0$ when dealing with the operations $s_\omega$ (for $\omega=0$, $s_\omega$ is the identity, which is of no interest here). An element $a \in MU_\ast(X)$ is called primitive if $s_\omega(a)=0$ for all $\omega$.

We may regard $\pi_\ast(MU)=MU_\ast$ as a submodule of $H_\ast(MU)$ by means of the Hurewicz homomorphism. If $H$ denotes the integral Eilenberg-Mac Lane spectrum and $S$ the sphere spectrum, then

$$H_\ast(MU)/MU_\ast \approx MU_\ast(H/S).$$

Let $N_1=\{a \in H_\ast(MU); s_\omega a \in MU_\ast \text{ for all } \omega\};$ then $N_1/MU_\ast$ can be identified with the primitive elements in $MU_\ast(H/S)$. Panov gives a determination of $N_1$.

Theorem (Panov). Let $n \geq 0$. Then $(N_1)_{2n}/MU_{2n}$ is a cyclic group. Moreover,

(a) for $n$ odd, it has order 2 with generator $CP(1)^n/2$;
(b) for $n=2$, it has order 12 with generators $CP(1)^3/4$ and $CP(2)/3$;
(c) for $n$ even and $n \geq 2$, it has $p$-torsion iff $n \equiv 0 \mod(p-1)$. For such $p$ write $k=n/(p-1)=p^m l$, $(p,l)=1$. If $p$ is odd, the $p$-component has order $p^{m+1}$ and generator $CP(p-1)^k/p^{m+1}$. If $p=2$, the 2-primary part has order

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2^{m+2} \text{ and generator}

\[ CP(1)/2^{m+2} + H_3CP(1)^{n-3}/2 \]

where \( H_3 \) is a suitable generator of \( MU_6 \) (e.g., \( H_3 = 2CP(3) + H_{2,2} \)).

Remarks 1. In fact Buhštaber ([1], [2]) introduces a filtration

\[ MU_* = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq H_*(MU) \]

where \( N_{i+1}/N_i \) is the module of primitive elements in \( H_*(MU)/N_i \). This explains the notation.

2. Let \( s_\omega \) denote characteristic numbers in \( K \)-theory, in particular when \( \omega \) is empty \( s_\omega = Td \), the Todd genus. The following is Panov's Proposition 6.

**Proposition.** \( N_1 \) consists of those elements in \( H_*(MU) \) for which \( s_\omega \) is integral for every \( \omega > 0 \).

Panov overlooked the following simple proof. First, if \( a \in N_1 \) and \( \omega > 0 \), then \( s_\omega(a) = Td s_\omega(a) \in \mathbb{Z} \) as desired. Conversely, let \( a \in H_*(MU) \) have all \( s_\omega(a) \) integral for \( \omega > 0 \). To show \( a \in N_1 \), fix \( \omega_0 > 0 \); we will show \( s_{\omega_0}(a) \in MU_* \). In view of the Hattori-Stong theorem, it suffices to show \( s_{\omega_1} \circ s_{\omega_0}(a) \in \mathbb{Z} \) for all \( \omega_1 \geq 0 \). This is the same as \( Td(s_{\omega_1} \circ s_{\omega_0}(a)) \). Since \( s_{\omega_1} \circ s_{\omega_0} \) is a linear combination of \( s_\omega \)'s with \( \omega > 0 \) ([4], [5]), this is clear.

3. One concludes that Panov could just as well use \( K \)-theory characteristic numbers. For example, at no point are the operations \( s_\omega \) composed.

4. Panov does not pick his generator \( H_3 \) properly, since \( s_3(H_{1,3}) = 0 \) and not \(-4\). A suitable choice is \( H_3 = 2CP(3) + H_{2,2} \), since \( s_3(H_3) = 2 \) and \( H_3 \) agrees with \( H_{2,2} \) mod 2 (see Proposition 3 and the preceding paragraph in [6]).

5. The evident \( BP \) analogue of Panov's theorem is valid. One simply replaces \( CP(p-1) \) with \( v_1 \), and \( H_3 \) with \( v_2 \) in the interesting case \( p = 2 \) and \( n \) even, \( n > 2 \).

We now outline our proof of Panov's theorem.

(a) Show \( (N_1)_{2n}/MU_{2n} \) is cyclic and has \( p \)-torsion iff \( n \equiv 0 \mod (p-1) \).

(b) For such \( p \), write \( n = (p-1)k \) and \( k = p^m l \) with \( (p, l) = 1 \). Show that \( CP(p-1)^k/p^{m+1} \) lies in \( (N_1)_{2n} \), and unless \( p = 2 \) and \( n \) is even and \( n > 2 \) it generates the \( p \)-component.

(c) If \( n \) is even and \( n > 2 \), show that \( P = CP(1)^n/2^{m+2} + H_3CP(1)^{n-3}/2 \) lies in \( (N_1)_{2n} \) and generates the 2-component of \( (N_1)_{2n}/MU_{2n} \).

By proving (a) first, one avoids the need for Panov's splittings. (b) is included in Proposition 4 of [6]. Our main contribution is in the proof of (c).

At this point it is convenient to follow [6] and choose generators \( \{H_n\} \) for \( MU_* \), \( H_n \in MU_{2n} \), such that \( H_{p-1} = CP(p-1) \) and, if \( k > 0 \),
$H_{p^{k+1}-1}$ is a nonzero multiple mod $p$ of Stong's hypersurface ([3], [7])

$$H_{p^k} \times \cdots \times H_{p^k} \subset CP(p^k) \times \cdots \times CP(p^k)$$

(where $p^k$ occurs $p$ times). In particular, $H_3 \equiv H_{2,2} \mod 2$.

For a fixed prime $p$ and $a \in N_1$, we say that $a$ has type $\omega_0$ if $s_{\omega_0}(a) \not\equiv 0 \mod p$ and $s_{\omega_0}(a) \equiv 0 \mod p$ if $\omega > \omega_0$. Recall that one puts $\omega > \omega_0$ if $|\omega| > |\omega_0|$, or if $|\omega| = |\omega_0|$ and $\omega$ involves fewer terms than $\omega_0$; see [3], [7]. For example, $H_{p-1}$ has type 0; $H_{p^{k+1}-1}$ has type $(p^k-1, \cdots, p^k-1)$ if $k > 0$ (where $p^k-1$ occurs $p$ times); and $H_n$ has type $(n)$ if $n+1$ is not a power of $p$. For the $H_{p^{k+1}-1}$ this is due to Stong ([3], [7]) if one computes with $K$-theory characteristic numbers. The Chern-Dold character makes it possible to carry over the computation to $MU$-characteristic numbers; see Proposition 2 of [6].

Notice that one has $s_0CP(1)^n = 0$ if $\omega \not\equiv (1, \cdots, 1)$. As is customary, write $s_\omega = s^{kA_1}$ if $\omega = (1, \cdots, 1)$ with $k$ 1's. Then we have

$$s^{kA_1}CP(1)^n = \left(\begin{array}{l} n \\ k \end{array}\right) 2^k CP(1)^{n-k},$$

and so it is convenient to notice the following property of binomial coefficients. We write $s_p = s_{kA_1}$ if $\omega = (1, \cdots, 1)$ with $k$ 1's. Then we have

**Lemma** Let $n = p^m l$, $(l, p) = 1$, where $p$ is a prime. If $k \leq p^n$ then $v_p(\omega) = m - v_p(k)$.

**Proof.** Simply notice that

$$\left(\begin{array}{l} n \\ k \end{array}\right) = \frac{p^m l}{k} \prod_{i=1}^{k-1} \frac{p^m l - i}{i}$$

and that $v_p(p^m l - i) = v_p(i)$ in this range.

**Proof of (a).** We show that in $(N_1)_{2n}/MU_{2n}$ the elements of order $p$ must be multiples of $CP(p-1)^k/p$, where $k = n/(p-1)$ is assumed to be integral. For let $a \in (N_1)_{2n}$, $pa \in MU_{2n}$, $a \notin MU_{2n}$. Write $a = \lambda CP(p-1)^k + \sum \lambda_i H_i$ with $\lambda$ and $\lambda_i$ rational (the $H_i$ are monomials in the generators of $MU_*$). Since $pa \in MU_{2n}$, $p\lambda$ and $p\lambda_i$ are integral. An argument with types (compare [3, §14]) now shows that $p\lambda_i \equiv 0 \mod p$, hence all $\lambda_i \in Z$. Since

$$a = p\lambda (CP(p-1)^k/p) + \sum \lambda_i H_i,$$

$a$ has the desired form.

**Proof of (b).** A computation using the lemma shows easily that, for $\omega > 0$, $s_\omega(CP(p-1)^k) \equiv 0 \mod p^{m+1}$, so $CP(p-1)^k/p^{m+1}$ belongs to $N_1$.  

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For \( p > 2 \) one computes that \( CP(p-1)^k/p^{m+1} \) has type \( (p-1) \). This also holds for \( p = 2 \) and \( m = 0 \) (i.e., \( n \) odd). For \( p = 2 \) and \( m > 0 \) one finds that \( CP(1)^n/2^{m+1} \) has type \( (1, 1) \); explicitly, \( s_1(CP(1)^n/2^{m+1}) \equiv CP(1)^{n-1} \mod 2 \), \( s_{1,1}(CP(1)^n/2^{m+1}) \equiv CP(1)^{n-2} \mod 2 \), and \( s_\omega(CP(1)^n/2^{m+1}) \equiv 0 \mod 2 \) if \( \omega > (1, 1) \). These observations make up Proposition 4 of [6], hence we do not offer more details.

**Proof of (c).** Let \( p = 2 \), \( n \) even and \( n > 2 \). Write \( n = 2^m l \) with \( l \) odd, hence \( m > 0 \). Using the \( MU \)-characteristic numbers of \( H_{2,2} \) listed in Proposition 3 of [6], we learn that \( s_1 H_3 \equiv CP(1) \mod 2 \), \( s_{1,1} H_3 \equiv CP(1)^2 \mod 2 \) and \( s_\omega H_3 \equiv 0 \mod 2 \) if \( \omega > (1, 1) \). It follows that

\[
s_\omega [CP(1)^n/2^{m+1} + H_3 CP(1)^{n-3}] \equiv 0 \mod 2 \]

for all \( \omega > 0 \), hence \( P = CP(1)^n/2^{m+2} + H_3 CP(1)^{n-3}/2 \) belongs to \( N_4 \).

The next task is to determine the type of \( P \). We establish a little more:

(d) \( s_3(P) \equiv 0 \mod 2 \); if \( m = 1 \) then \( P \) has type \( (3) \), and if \( m > 1 \) then \( P \) has type \( (1, 1, 1, 1) \).

To show that (d) implies (c), we must show that it is impossible to have \( s_\omega(P) \equiv s_\omega(a) \mod 2 \) for all \( \omega > 0 \) with \( a \in MU_{2n} \). By an argument with types, this can happen only if \( P \) has type \( (1, 1, 1, 1) \), i.e., \( m > 1 \), and then \( a \) must be a linear combination of \( CP(1)^{n-6} H_3^2 \), \( CP(1)^{n-5} H_3 \), \( CP(1)^{n-2} CP(2) \), and \( CP(1)^n \). But then we would have \( s_3(a) \equiv 0 \mod 2 \), which violates (d).

**Proof of (d)** First of all one computes directly that \( s_3(P) = CP(1)^{n-3} \), hence \( s_3(P) \equiv 0 \mod 2 \). Thus it remains to consider \( s_\omega(P) \) with \( \omega > (3) \).

If \( \omega \neq (1, \cdots, 1) \), then

\[
s_\omega(P) = \frac{1}{2} s_{\omega'}(H_3 CP(1)^{n-3}).
\]

Since \( s_\omega(H_3) = 0 \) for dimensional reasons, \( s_{\omega'}(H_3) \equiv 0 \mod 2 \) if \( \omega' \neq (1, \cdots, 1) \) and \( s_{\omega'}(CP(1)^{n-3}) \equiv 0 \) if \( \omega'' \neq (1, \cdots, 1) \) it follows easily that \( s_\omega(P) \equiv 0 \mod 2 \).

So we need only consider \( s^{k \Delta_1}(P) \) for \( k \geq 4 \). One first shows that

\[
s^{k \Delta_1}(CP(1)^n + 2^{m+1} H_3 CP(1)^{n-3}) \equiv \binom{n}{k} 2^k CP(1)^{n-k} \mod 2^{m+3},
\]

hence one wants to know when \( v_2(\binom{2m}{k}) + k \geq m+3 \). E.g., if \( m = 1 \) we want to know when \( v_2(\binom{2}{k}) + k \geq 4 \); since \( k \geq 4 \) this always holds, hence \( s^{k \Delta_1}(P) \equiv 0 \mod 2 \) for \( k \geq 4 \) when \( m = 1 \). This proves (d) for \( m = 1 \).

Finally let \( m > 1 \). Note that we may assume \( 1 \leq k \leq m+2 \leq 2^m \). Then the lemma implies that

\[
v_2(\binom{2m}{k}) = m - v_2(k).
\]
Hence \( v_2(2^m) + 3 \) iff \( k \geq m + 3 \), and this fails for \( k = 4 \), hence \( s_1,1,1,1(M) \neq 0 \mod 2 \); but it holds for \( k > 4 \), which proves (d) for \( m > 1 \).

Q.E.D.

REFERENCES


DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903