

## PRODUCTS OF RC-PROXIMITY SPACES

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**ABSTRACT.** It is shown that the proximity product of RC-proximity spaces need not be an RC-proximity space. This answers a question of Harris. A positive answer would have implied that the product of regular-closed spaces is regular-closed.

1. **Introduction.** D. Harris [4] has characterized those spaces which can be densely embedded in a regular-closed space as those regular spaces which admit a compatible generalized proximity, which he called an RC-proximity. In [4] he asked five questions. The first and third have since been answered by Sharma and Naimpally [9] while Hunsaker and Sharma have recently solved the second by giving necessary and sufficient conditions under which a map from an RC-regular space to a regular-closed space can be extended to the Harris extension. (The following variant of Harris' second problem is still open: characterize those topological spaces  $X$  for which any continuous function from  $X$  to any regular-closed space can be extended to the Harris extension.) The purpose of this note is to give a negative answer to part of the fourth problem which asked if the product of RC-proximities with the usual definition of product proximity (see Čech [2] or Willard [11]) is an RC-proximity. A positive answer to this question would have implied that the product of regular-closed spaces is regular-closed [1, problem 5]. We shall prove that if  $(X, \delta)$  is an absolutely closed RC-proximity space which is not compact, then there is a compact Hausdorff space  $Y$  for which  $X \times Y$  with the product structure is not an RC-proximity space. Our proof will make use of a recent result of Willard [10].

Proximity, unmodified, will refer to the usual proximity. A proximally continuous function is called a  $p$ -map. Otherwise, all notation and terminology will follow Harris [4].

### 2. Preliminary definitions and lemmas.

2.1. **DEFINITION.** (See Willard [11].) If  $\{(X_\alpha, \delta_\alpha)\}_\alpha$  is a collection of sets with binary relations then the product structure  $\delta$  on  $X = \prod X_\alpha$  is

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defined as follows:

$A \delta B$  iff whenever  $A=A_1 \cup \dots \cup A_n$  and  $B=B_1 \cup \dots \cup B_m$  then there is some  $A_i$  and  $B_j$  for which  $\pi_\alpha(A_i) \delta_\alpha \pi_\alpha(B_j)$  for all  $\alpha$ .

When each  $(X_\alpha, \delta_\alpha)$  is a proximity space then  $\delta$  is the product proximity and is the categorical product.

2.2. LEMMA. *If  $\{(X_\alpha, \delta_\alpha)\}_\alpha$  is a collection of  $R$ -proximity spaces, then  $X=\prod X_\alpha$  with the product structure  $\delta$  is an  $R$ -proximity space.  $\delta$  is the coarsest  $R$ -proximity on  $X$  for which each  $\pi_\alpha$  is a  $p$ -map.*

PROOF. It follows from Leader's result in [6] that P1–P4 of Harris' definition are satisfied and it is sufficient to verify P5. Let  $x \not\delta A$ . We shall find a set  $B \subseteq X$  for which  $x \not\delta (X-B)$  and  $B \not\delta A$ . This will clearly satisfy P5. Since  $x \not\delta A$ , there is a finite decomposition of  $A$ ,  $A=A_1 \cup \dots \cup A_n$  such that for each  $i$ , there is some  $\alpha_i$  for which  $\pi_{\alpha_i}(x) \not\delta_{\alpha_i} \pi_{\alpha_i}(A_i)$ . Since each  $X_{\alpha_i}$  is an  $R$ -proximity space, for each  $i$  there is some  $K_i \subseteq X_{\alpha_i}$  such that  $\pi_{\alpha_i}(x) \not\delta_{\alpha_i} (X_{\alpha_i}-K_i)$  and  $K_i \not\delta_{\alpha_i} \pi_{\alpha_i}(A_i)$ . Let  $B=\bigcap_i \pi_{\alpha_i}^{-1}(K_i)$ . Then  $X-B=\bigcup_i \pi_{\alpha_i}^{-1}(X_{\alpha_i}-K_i)$ . Now,  $x \not\delta \pi_{\alpha_i}^{-1}(X_{\alpha_i}-K_i)$  for each  $i$  implies that  $x \not\delta (X-B)$ , and since  $A_i \not\delta \pi_{\alpha_i}^{-1}(K_i)$ ,  $A_i \not\delta B$  for each  $i$ , so that  $A \not\delta B$ . That  $\delta$  is the coarsest such structure follows easily from the fact that if  $Z$  is an  $R$ -proximity space then  $f:Z \rightarrow X$  is a  $p$ -map iff  $\pi_\alpha \circ f$  is a  $p$ -map for all  $\alpha$ .

2.3. LEMMA (WILLARD [10]). *If  $X$  is a topological space such that for every compact Hausdorff space  $Y$ ,  $\pi_Y: X \times Y \rightarrow Y$  is a closed map, then  $X$  is compact.*

The next lemma is a special case of Theorem 7 of [6].

2.4. LEMMA. *Let  $(X_1, \delta_1)$  and  $(X_2, \delta_2)$  be  $R$ -proximity spaces and let  $\delta$  be the product  $R$ -proximity. Then  $y \delta_i \pi_i(A)$  implies  $\pi_i^{-1}(y) \delta A$  for  $i=1$  or  $2$ .*

PROOF. We shall prove this for  $i=2$ . Let  $y \delta_2 \pi_2(A)$  and assume  $\pi_2^{-1}(y) \not\delta A$ . Then there are finite decompositions of  $A$  and  $\pi_2^{-1}(y)$ ,  $A=A_1 \cup \dots \cup A_n$  and  $\pi_2^{-1}(y)=C_1 \cup \dots \cup C_m$  and for each  $A_i$  and  $C_j$  either  $\pi_1(A_i) \delta_1 \pi_1(C_j)$  or  $\pi_2(A_i) \delta_2 \pi_2(C_j)$ . Now, some member of the decomposition of  $A$ , say  $A_k$ , must satisfy  $\pi_2(A_k) \delta_2 y$ . Since  $\pi_2^{-1}(y) \not\delta A$ , it follows from the definition of the product that  $\pi_1(A_k) \not\delta_1 \pi_1(C_j)$  for all  $j$ . But then  $\pi_1(A_k)$  is separated from  $\bigcup_j \pi_1(C_j)$  and  $\bigcup_j \pi_1(C_j)=\pi_1(\bigcup_j C_j)=\pi_1 \pi_2^{-1}(y)=X_1$ . This is clearly impossible.

### 3. Main theorem.

3.1. THEOREM. *If  $(X_1, \delta_1)$  is an absolutely closed RC-proximity space which is not compact then there exists a compact Hausdorff space  $X_2$  with*

its unique compatible proximity  $\delta_2$  for which  $X_1 \times X_2$  with the product  $R$ -proximity is not an RC-proximity.

PROOF. Let  $(X_1, \delta_1)$  be as in the statement of the theorem. If the result does not hold then for each compact Hausdorff space  $X_2$ , the product  $R$ -proximity  $\delta$  on  $X_1 \times X_2$  is an RC-proximity. By the theorem of Scarborough and Stone [8],  $X_1 \times X_2$  with the product topology is regular-closed and hence by [4] has exactly one RC-proximity which induces its topology. This RC-proximity is given by  $A \delta_0 B$  iff  $\bar{A} \cap \bar{B} \neq \emptyset$ . It follows from our assumption that  $\delta = \delta_0$ . We claim that  $\pi_2: X_1 \times X_2 \rightarrow X_2$  is closed. Let  $F$  be a closed subset of  $X_1 \times X_2$  and let  $y \in \text{Cl}((\pi_2(F)))$ . That is, let  $y \delta_2 \pi_2(F)$ . By Lemma 2.4,  $\pi_2^{-1}(y) \delta F$ , so that  $\pi_2^{-1}(y) \delta_0 F$ . Since both  $\pi_2^{-1}(y)$  and  $F$  are closed,  $\pi_2^{-1}(y) \cap F \neq \emptyset$ , from which it follows that  $y \in \pi_2(F)$  and  $\pi_2(F)$  is closed. Since this is true for any compact Hausdorff space  $X_2$ , Lemma 2.3 implies that  $X_1$  is compact. This contradiction establishes the theorem.

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