THE CODIMENSION OF THE BOUNDARY
OF A LATTICE IDEAL[1]

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ABSTRACT. In a compact connected topological lattice of finite codimension n, the boundary of a proper principal ideal has codimension less than n. It follows that the boundaries of proper intervals also have codimension less than n.

In a topological lattice L the boundary of a principal (dual) ideal is a join (meet) subsemilattice of L [1, Lemma 5]. The purpose of this paper is to show that this semilattice has codimension less than n whenever L is a compact, connected topological lattice of codimension n. Note that such a lattice has breadth n [5, Corollary 2.4].

If S is a semilattice and $x \in S$, then $M(x)=\{y \in S : x \leqq y\}$; the set $L(x)$ is defined dually; if $x \leqq y$, then $[x, y]=M(x) \cap L(y)$. The interior and the closure of A are denoted by $A°$ and A^* respectively. The boundary of A is $F(A)=A^* \backslash A°$. If $a \in S$ a topological semilattice and $x \in F(M(a))$, then $[a, x] \subset F(M(a))$.

LEMMA. *Let L be a compact, connected topological lattice of finite codimension. If $a \in L$, then $F(M(a))$ is locally connected.*

PROOF. Let $x \in F(M(a))$ and let U be an open subset of $F(M(a))$ which contains x. It is known that L has a basis of open convex sublattices [6, Theorem 5]. Hence we may choose an open convex sublattice V containing x such that $V \cap F(M(a)) \subset U$. Let $p, q \in V \cap F(M(a))$; then the connected set $[p \wedge q, p] \cup [p \wedge q, q]$ is contained in $V \cap F(M(a))$. Thus $V \cap F(M(a))$ is a connected open subset of $F(M(a))$ which contains x and which is contained in U. Therefore $F(M(a))$ is locally connected.

We shall state and prove the main result in terms of principal dual ideals.

THEOREM. *Let L be a compact, connected topological lattice of finite codimension n and let $a \in L \backslash \{0, 1\}$. Then the codimension of $F(M(a))$ is less than n.*

PROOF. Since the result clearly holds for $n \leq 1$, we shall assume $n > 1$.

We shall first show that $x \in F(M(a))$ implies $[a, x]$ has codimension less than n. Let $x \in F(M(a))$ and let $U(a) = L \setminus [L(a) \cup M(a)]$. If $U(a) = \varnothing$, then a is a cut point of L [2, Theorem 1]. Thus $a = x$ and $[a, x] = \{a\}$ has codimension 0.

Suppose $U(a) \neq \varnothing$ and $x \neq a$. We shall construct an increasing net in $L(x) \cap U(a)$ which converges to x. Since $x \notin L(a)$ we may choose an open set V containing x such that $V \cap L(a) = \varnothing$. Now $x \in F(M(a))$ implies $V \cap L \setminus M(a) \neq \varnothing$. Note that $L \setminus M(a) = [L(a) \cup U(a)] \setminus \{a\}$ so that $V \cap U(a) \neq \varnothing$. Thus we may choose a net $\{x'_\alpha\}_{\alpha \in \Delta} \subset U(a)$ which converges to x. Let $y_\alpha = x \wedge x'_\alpha$ for all $\alpha \in \Delta$. Then $\{y_\alpha\}_{\alpha \in \Delta} \subset L(x)$ and converges to x. If $\{y_\alpha\}_{\alpha \in \Gamma} \subset L(a)$ for any cofinal subset $\Gamma \subset \Delta$, then $\{(y_\alpha, a)\}_{\alpha \in \Gamma}$ converges to $(x, a) \in L \times L$. Since the graph of $\leq$ is a closed subset of $L \times L$, then $x \leq a$. But $x \in M(a)$ so that $x = a$ contrary to the choice of $x \neq a$. Hence we may assume $y_\alpha \notin L(a)$ for all $\alpha \in \Delta$. If $y_\alpha \in M(a)$, then $a \leq y_\alpha = x \wedge x'_\alpha \leq x'_\alpha$ contrary to $a \not\leq x'_\alpha$. Thus $\{y_\alpha\}_{\alpha \in \Delta} \subset L(x) \cap U(a)$.

Let $x_\alpha = \bigwedge_{\beta \geq \alpha} y_\beta$ for each $\alpha \in \Delta$. Clearly $\{x_\alpha\}_{\alpha \in \Delta}$ is an increasing net contained in $L(x)$. Let U be an open set containing x. There exists a closed sublattice $V \subset U$ such that $x \in V^\circ$ [6, Theorem 5]. Since $\{y_\alpha\}_{\alpha \in \Delta}$ converges to x, there exists $\beta \in \Delta$ such that $\alpha \geq \beta$ implies $y_\alpha \in V^\circ$. Hence $x_\gamma = \bigwedge_{\alpha \geq \gamma} y_\alpha \in V$ for all $\gamma \geq \beta$. Thus $\{x_\alpha\}_{\alpha \in \Delta}$ converges to x. That $\{x_\alpha\}_{\alpha \in \Delta} \subset U(a)$ follows just as did the fact that $\{y_\alpha\}_{\alpha \in \Delta} \subset U(a)$. Therefore $\{x_\alpha\}_{\alpha \in \Delta}$ is the required net.

For each $\alpha \in \Delta$, the interval $[a, a \vee x_\alpha]$ has breadth less than n [7, Lemma 1.1]. Since $\{x_\alpha\}_{\alpha \in \Delta}$ is increasing, $\{[a, a \vee x_\alpha]\}_{\alpha \in \Delta}$ is a chain. Therefore $\bigcup_{\alpha \in \Delta} [a, a \vee x_\alpha]$ has breadth less than n; consequently the breadth of $(\bigcup_{\alpha \in \Delta} [a, a \vee x_\alpha])^*$ is less than n. Since the interval $[a, a \vee x_\alpha] = (a \vee x_\alpha) \wedge M(a)$, it follows that $[a, x] = (\bigcup_{\alpha \in \Delta} [a, a \vee x_\alpha])^*$ [3, Theorem 3]. Thus $[a, x]$ has breadth less than n, and since $[a, x]$ is a compact, connected topological lattice, its breadth and its codimension are equal.

The semilattice $F(M(a))$ is a Lawson semilattice, i.e. has a neighborhood basis of subsemilattices [5, Theorem 1.1]. A. Y. W. Lau has shown that any compact, connected, locally connected Lawson semilattice S contains a point x for which the codimension of S and the codimension of $L(x)$ are equal [4, Lemma 5.2]. Thus for some $x \in F(M(a))$ the codimension of $[a, x]$ and the codimension of $F(M(a))$ are equal.

COROLLARY. *Let L be a compact connected topological lattice of finite codimension n. If $a, b \in L$ and $a < b$, then the codimension of $F([a, b])$ is less than n. Thus L has a basis of open sets whose boundaries have codimension less than n.*

PROOF.   If $a=0$ or $b=1$, then $[a, b]=L(b)$ or $[a, b]=M(a)$. Thus we may assume $a\neq0$ and $b\neq1$. It is easy to see that $F([a, b])$ is a closed subset of $F(M(a))\cup F(L(b))$. By the theorem each of these sets has co-dimension less than n. Thus the codimension of $F([a, b])$ is less than n.

Since L has a basis of neighborhoods at each point consisting of intervals $[a, b]$ [6, Theorem 5], the collection of interiors of these intervals is the desired basis.

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