ON SUM-FREE SUBSEQUENCES[1]

DAVID G. CANTOR

ABSTRACT. A sequence of real numbers is said to be *sum-free* if no number of the sequence is the sum of distinct elements of the same sequence. In this paper we show that a sequence S of n positive real numbers has a sum-free subsequence containing at least $(2n)^{1/2} - \log_2(4n)$ elements.

Choi [1], studying a problem of Erdös [2], has proven that a sequence of n positive real numbers has a sum-free subsequence of $\geq (36/35)n^{1/2}$ elements. The purpose of this paper is to show that the constant $36/35$ can be improved to $2^{1/2} - \varepsilon$. In fact, more precisely, we have

THEOREM. *A sequence S of n positive real numbers has a sum-free subsequence containing* $\geq (2n)^{1/2} - \log_2(4n)$ *elements.*

PROOF. Let a_1 be the least element of S; let a_2 be the least element of S which is $\geq 2a_1$, and inductively choose a_{i+1} to be the least element of S which is $\geq 2a_i$. This yields a finite sequence $a_1 < a_2 < \cdots < a_m$ and every element of S is $< 2a_m$. For $1 \leq i \leq m-1$, let T_i be the subsequence of S consisting of those elements of S which are $\geq a_i$ and $< a_{i+1}$; let T_m be the subsequence of S consisting of those elements of S which are $\geq a_m$. Denote by t_i the cardinality of T_i. Then S is the disjoint union of the T_i and $\sum_{i=1}^{m} t_i = n$. Now

$$\sum_{i=1}^{m} (t_i + m - i) = n + \frac{m(m-1)}{2}$$

or

$$\frac{1}{m} \sum_{i=1}^{m} (t_i + m - i) = \frac{n}{m} + \frac{m}{2} - \frac{1}{2}.$$

Minimizing over m shows that $(1/m) \sum_{i=1}^{m} (t_i + m - i) \geq (2n)^{1/2} - \frac{1}{2}$. Thus there is an index j such that $t_j + m - j \geq (2n)^{1/2} - \frac{1}{2}$. Define $k = j + 1 + \lceil \log_2 j \rceil$. (Here and throughout $[x]$ denotes the least integer $\geq x$). Put $V = \{a_k, a_{k+1}, \cdots, a_m\}$; V is a sum-free sequence because its elements grow so rapidly that each element of V is greater than the sum of all smaller

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elements of V. The sequence T_j is sum-free because each element of T_j is $\geq a_j$ and $< 2a_j$, and as a result the sum of any two elements of T_j is greater than any element of T_j. Finally,

the sum of all the elements of $T_j < t_j \cdot 2a_j$

$$= 2^{(1+\log_2 t_j)} a_j \leq 2^{1+\lceil \log_2 t_j \rceil} a_j$$

$$\leq a_{j+1+\lceil \log_2 t_j \rceil} = a_k \quad \text{if } k \leq m.$$

It is immediate that if $a_i \in V$, then a_i is greater than the sum of all smaller elements of $T_j \cup V$ and hence $T_j \cup V$ is a sum-free subsequence of S with

number of elements $\geq t_j + m - k + 1 = t_j + m - j - [\log_2 j]$

$$\geq (2n)^{1/2} - \tfrac{1}{2} - \log_2(2j) \geq (2n)^{1/2} - \tfrac{1}{2} - \log_2(2n)$$

$$\geq (2n)^{1/2} - \log_2(4n).$$

References

1. S. Choi, *The largest sum-free subsequence from a sequence of n numbers*, Proc. Amer. Math. Soc. **39** (1973), 42–44.

2. P. Erdös, *Extremal problems in number theory*, Proc. Sympos. Pure Math., vol. 8, Amer. Math. Soc., Providence, R.I., 1965, pp. 181–189. MR **30** #4740.

Department of Mathematics, University of California, Los Angeles, California 90024