

MULTIPLE POINTS OF TRANSIENT RANDOM WALKS

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ABSTRACT. We determine the asymptotic behavior of the expected numbers of points visited exactly j times and at least j times in the first n steps of a transient random walk on a discrete Abelian group. We prove that the strong law of large numbers holds for these multiple point ranges.

Let X_1, X_2, \dots be a sequence of independent identically distributed random variables (taking values in an arbitrary countable Abelian group), and let $S_1 = X_1, S_2 = X_1 + X_2, S_3 = X_1 + X_2 + X_3 + \dots$ be the sequence of partial sums of $\{X_n\}$. The range R_n of the random walk associated with X_1, X_2, \dots is the number of distinct values assumed by the finite sequence of partial sums S_1, S_2, \dots, S_n . Dvoretzky and Erdős [1] (and by different means Spitzer, Kesten and Whitman [3, pp. 38–40]) have shown that for any random walk $\lim_{n \rightarrow \infty} E(R_n/n) = 1 - F$, where F is the probability that $\exists n \ni S_n = 0$, and, in fact, the strong law of large numbers applies to the range in the sense that $\lim_{n \rightarrow \infty} R_n/n = 1 - F$ a.s. (The latter result is of interest principally when the walk is transient, i.e. $F \neq 1$; Dvoretzky and Erdős obtain the stronger $\lim_{n \rightarrow \infty} R_n/E(R_n) = 1$ a.s. for simple random walk in the plane.)

We obtain analogous results concerning the number of distinct values assumed exactly j times and at least j times by the sequence S_1, S_2, \dots, S_n .

We introduce the notation:

R_n^j = the # of distinct values occurring at least j times in the sequence S_1, \dots, S_n .
 $R_n^{(j)}$ = the # of distinct values occurring exactly j times in the sequence S_1, \dots, S_n .

Our main result is:

THEOREM. For any random walk and any j , $\lim_{n \rightarrow \infty} (R_n^j/n) = (1 - F)F^{j-1}$ a.s., and $\lim_{n \rightarrow \infty} (R_n^{(j)}/n) = (1 - F)^2 F^{j-1}$ a.s.

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Note. (1) For recurrent walks this result is a direct consequence of the result for $j=1$, and of no interest.

(2) The corresponding result for simple walk on $Z^n, n \geq 1$, is stated by Erdős and Taylor [2]. Their method is entirely different and applies only to walks on Z^n .

PROOF. The proof is inductive. The result is already known for R_n^1 ; so all we need do is establish the induction step: $R_n^j/n \rightarrow (1-F)F^{j-1}$ a.s. implies $R_n^{(j)}/n \rightarrow (1-F^2)F^{j-1}$ a.s. and $R_n^{j+1}/n \rightarrow (1-F)F^{j-1}$ a.s.

We assume then that $R_n^j/n \rightarrow (1-F)F^{j-1}$ a.s. and turn to the first order of business, the determination of the behavior of $E(R_n^{(j)})/n$ and $E(R_n^{j+1})/n$ as $n \rightarrow \infty$. To this end we write:

$$\begin{aligned} Z_k &= 1, && \text{if there are exactly } j-1 \text{ indices } l < k \ni S_l = S_k, \\ &= 0, && \text{otherwise.} \\ Y_{k,n} &= 1, && \text{if there are exactly } j-1 \text{ indices } l < k \text{ such that} \\ &&& S_l = S_k \text{ and } S_{k+1} \neq S_k, S_{k+2} \neq S_k, \dots, S_n \neq S_k, \\ &= 0, && \text{otherwise.} \end{aligned}$$

We have $R_n^{(j)} = \sum_{k=1}^n Y_{k,n}$ so that $E(R_n^{(j)}) = \sum_{k=1}^n E(Y_{k,n})$. Also, X_{k+1}, \dots, X_n , are independent of Z_k , hence

$$\begin{aligned} E(Y_{k,n}) &= E(Z_k)P(X_{k+1} \neq 0, X_{k+1} + X_{k+2} \neq 0, \dots, X_{k+1} + \dots + X_{k+n} \neq 0) \\ &= E(Z_k)P(S_1 \neq 0, \dots, S_{n-k} \neq 0). \end{aligned}$$

But also $R_n^j = \sum_{k=1}^n Z_k$ and, in view of the induction hypothesis and the Lebesgue dominated convergence theorem,

$$(1) \quad \sum_{k=1}^n E(Z_k)/n = E(R_n^j)/n \rightarrow (1-F)F^{j-1}.$$

Furthermore, if we write, $f_k = P(S_1 \neq 0, S_2 \neq 0, \dots, S_{k-1} \neq 0, S_k = 0)$ we have $P(S_1 \neq 0, \dots, S_{n-k} \neq 0) = 1 - \sum_{i=1}^{n-k} f_i$ and, as $n \rightarrow \infty$,

$$(2) \quad 1 - \sum_{i=1}^{n-k} f_i \rightarrow 1 - F.$$

A straightforward argument then yields

$$(3) \quad E(R_n^{(j)})/n \rightarrow (1-F)^2 F^{j-1},$$

and the further observation that $R_n^{j+1} = R_n^j - R_n^{(j)}$ gives

$$(4) \quad E(R_n^{j+1})/n \rightarrow (1-F)F^j.$$

To obtain the desired strong laws of large numbers we introduce the new sequences of random variables defined for $n=1, 2, \dots, k=1, 2, \dots$:

$$T_{k,n}^{(j)} = \text{the number of distinct values occurring exactly } j \text{ times in } S_{(k-1)n+1}, \dots, S_{kn},$$

$$T_{k,n}^{j+1} = \text{the number of distinct values occurring at least } j+1 \text{ times in } S_{(k-1)n+1}, \dots, S_{kn}.$$

For any m we have

$$(5) \quad R_{mn}^{(j)} \leq \sum_{k=1}^m T_{k,n}^{(j)} + E_{m,n},$$

$$(6) \quad R_{mn}^{j+1} \leq \sum_{k=1}^m T_{k,n}^{j+1} + E_{m,n},$$

where $E_{m,n}$ = the number of distinct sums which occur in at least two different blocks

$$\{S_{(k-1)n+1}, \dots, S_{kn}\}, \quad \{S_{(k'-1)n+1}, \dots, S_{k'n}\}, \quad k, k' \leq m.$$

We prove below that

$$\limsup_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} E_{m,n}/mn = 0,$$

and complete our proof of the theorem assuming the truth of this lemma.

We let $m \rightarrow \infty$ in (5) and obtain

$$(7) \quad \limsup_{m \rightarrow \infty} \frac{R_{mn}^{(j)}}{mn} \leq \lim_{m \rightarrow \infty} \frac{\sum_{k=1}^m T_{k,n}^{(j)}}{mn} + \lim_{m \rightarrow \infty} \frac{E_{m,n}}{mn}.$$

By the strong law of large numbers applied to the sequence of (bounded) independent identically distributed random variables $T_{k,n}$, $k=1, 2, \dots$,

$$(8) \quad \lim_{m \rightarrow \infty} \frac{\sum_{k=1}^m T_{k,n}^{(j)}}{mn} = \frac{1}{n} \lim_{m \rightarrow \infty} \frac{\sum_{k=1}^m T_{k,n}^{(j)}}{m} = \frac{E(T_{1,n}^{(j)})}{n} = \frac{E(R_n^{(j)})}{n}.$$

(7) yields a bound on $\limsup R_k^{(j)}/k$ as $k \rightarrow \infty$ through a subsequence consisting of multiples of n . The restriction of the mode of approach to a subsequence may be eliminated by observing that

$$R_k^{(j)}/k \leq \frac{R_{[k/n]_n + (k - [k/n]_n)}^{(j)}}{[k/n]_n} \leq R_{[k/n]_n}^{(j)}/[k/n]_n + 1/[k/n]$$

$$= R_{mn}^{(j)}/mn + o(1),$$

so that $\limsup_{k \rightarrow \infty} R_k^{(j)}/k \leq \limsup_{m \rightarrow \infty} R_{mn}^{(j)}/mn$ for any n .

Substituting (8) in (7), letting $n \rightarrow \infty$ and using (3) yields

$$(9) \quad \limsup_{k \rightarrow \infty} R_k^{(j)}/k \leq (1 - F)^2 F^{j-1} \quad \text{a.s.}$$

Trivial adjustments in the argument which led from (5) to (9) allow us to use (6) to conclude

$$(10) \quad \limsup_{k \rightarrow \infty} R_k^{j+1}/k \leq (1 - F)F^j.$$

But now we complete our argument with ease. For,

$$\liminf_{k \rightarrow \infty} R_k^{(j)}/k \geq \liminf_{k \rightarrow \infty} R_k^j/k - \limsup_{k \rightarrow \infty} R_k^{j+1}/k,$$

and in view of our induction hypothesis and (10) the right side is $\geq (1 - F)^2 F^{j-1}$. Finally

$$\lim_{k \rightarrow \infty} R_k^{j+1}/k = \lim_{k \rightarrow \infty} R_k^j/k - \lim_{k \rightarrow \infty} R_k^{(j)}/k = (1 - F)F^j.$$

PROOF OF LEMMA. We define the random variable ξ_i^n by

$$\begin{aligned} \xi_i^n &= 1, && \text{if } S_i \neq S_{i+1}, S_i \neq S_{i+2}, \dots, S_i \neq S_{\lfloor i/n \rfloor + 1}_n \\ &&& \text{but } \exists k > i \ni S_k = S_i \text{ or } n \mid i \text{ and } \exists k > i \ni S_k = S_i, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Clearly $0 \leq E_{m,n} \leq \sum_{i=1}^{mn} \xi_i^n$ (in fact, $E_{m,n} \leq \sum_{i=1}^{(m-1)n} \xi_i^n$). We now choose an n' corresponding to n so that $n' \rightarrow \infty$ as $n \rightarrow \infty$ and $n'/n \rightarrow 0$. We let

$$\begin{aligned} \eta_i^n &= 1, && \text{if } S_i \neq S_{i+1}, S_i \neq S_{i+2}, \dots, S_i \neq S_{i+n'} \\ &&& \text{but } \exists k > i + n' \ni S_k = S_i, \\ &= 0, && \text{otherwise.} \end{aligned}$$

The sequence of random variables η_i^n is stationary ergodic, hence

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \eta_i^n = E(\eta_i^n) = \sum_{n'} f_j,$$

by the ergodic theorem. Also $\xi_i^n \leq \eta_i^n$ unless $n \mid i$ or $[i/n] + n - i < n'$, so

$$0 \leq E_{m,n} \leq \sum_{i=1}^{mn} \xi_i^n + m \cdot n'.$$

Hence

$$0 \leq \limsup_{m \rightarrow \infty} \frac{E_{m,n}}{mn} \leq \lim_{m \rightarrow \infty} \frac{\sum_{i=1}^{mn} \xi_i^n}{mn} + \frac{n'}{n} = \sum_{n'} f_j + \frac{n'}{n}.$$

And as $n \rightarrow \infty$, $\sum_{n'} f_j + n'/n \rightarrow 0$ by our choice of n' .

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