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A SPECTRAL SEQUENCE FOR THE INTERSECTION OF SUBSPACE PAIRS

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ABSTRACT. A general-homology spectral sequence that generalizes the Mayer-Vietoris exact sequence is established between the intersection of a family of subspace pairs and the system of partial unions of the family. The basis of the construction is a topological analogue of the "bar construction" of homological algebra.

We shall show here that a finite family $\mathscr{P}=\{(X_i, A_i)|i \in I\}$ of subspace pairs[1] in a space X have, for each general homology theory h_*, a spectral sequence

$$\text{(a)} \qquad E^n_{1;j} \simeq \bigoplus_{Ns=n} h_j\Big(\bigcup_s X_i, \bigcup_s A_i\Big) \underset{n}{\Rightarrow} h_{j-n}\Big(\bigcap_I X_i, \bigcap_I A_i\Big)$$

($s \subset I$), Ns being (number of members in s)-1. This is just the spectral sequence of a cover with the roles of union and intersection interchanged. Its connection with the Mayer-Vietoris sequence will be examined below, and we shall derive from it the spectral sequence of the homology sheaf of X.

Construction of (a). Start with any finite set U (= the universe) that contains I as a subset, and define (using T to denote the based unit interval, while $\bigwedge_J Y = Y^J/\{\eta \in Y^J | * \in \eta J\}$ for based spaces Y and finite sets J)

$$\text{(b)} \qquad \begin{aligned} \nabla s &= \bigwedge_s \partial T \wedge \bigwedge_{U-s} T \qquad (s \subset U), \\ K &= \bigcup_{a \in U} \nabla\{a\} = \partial \nabla \varnothing, \\ C &= \{*\} \cup \bigcup_{a \in U-I} \nabla\{a\} = \bigwedge_I T \wedge \partial \bigwedge_{U-I} T, \\ M &= X \times C \cup \bigcup_{s \subset I} \bigcup_s A_i \times \nabla s, \\ L^n &= M \cup \bigcup_{s \subset I; Ns \geqq n} \bigcup_s X_i \times \nabla s \qquad (n \in \mathbf{Z}) \\ L &= L^0 = X \times C \cup \bigcup_{s \subset I} \bigcup_s X_i \times \nabla s. \end{aligned}$$

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[1] Assumed to be subcomplex pairs under some CW complex structure on X.

We require $I\neq\varnothing$ but permit $U=I$. *Caution.* Here ∂ is used in the context of based spaces, so $\partial\partial T=\{*\}$, not $\varnothing$. The formulas for the spectral sequence are as follows (as in [**1**, p. 108 ff.] for any filtered space):

$$E^n_{r;j}=\frac{\operatorname{Im}[h_p(L^n, L^{n+r})\to h_p(L^{n-r+1}, L^{n+r})]}{\operatorname{Im}[h_p(L^{n+1}, L^{n+r})\to h_p(L^{n-r+1}, L^{n+r})]}\quad (n, j\in \boldsymbol{Z};\ r=1, 2, \cdots)$$

where $p=j-n+NU$ and maps are induced by inclusion,

(c) $d^n_{r;j}=$ homomorphism $E^n_{r;j}\to E^{n+r}_{r;j+r-1}$ induced by ∂_p for the triple (L^n, L^{n+r}, L^{n+2r}),

$u^n_{r;j}=$ isomorphism $E^n_{r+1;j}\to \mathscr{H}^n_j E^*_{r;*}$ induced by $h_p(\subset)$ for the inclusion $(L^n, L^{n+r+1})\subset(L^n, L^{n+r})$.

Thus, $E^n_{\infty;j}=E^n_{r;j}$ for large r, $=F^nG_{j-n}/F^{n+1}G_{j-n}$, where

(c, Cont'd.)
$$G_q=h_{q+NU}(L, M)\qquad (q\in\boldsymbol{Z}),$$
$$F^nG_q=\operatorname{Im}[h_{q+NU}(L^n, M)\to h_{q+NU}(L, M)].$$

(Note that $G_*=F^0G_*\supset F^1G_*\supset\cdots\supset F^{NI+1}G_*=\{0\}$.)

Define also, for each $a\in U$ and $s\subset U$ containing a,

(d)
$$\nu(a):\boldsymbol{h}_q(\nabla s, \partial\nabla s)\xrightarrow[\text{excision}]{\cong}\boldsymbol{h}_q(\partial\nabla s', \partial\nabla s'-\nabla^\# s)\,(s'=s-\{a\}),$$
$$\mu(a):\boldsymbol{h}_{q+1}(\nabla s', \partial\nabla s')\xrightarrow[\cong]{\partial_{q+1}}\boldsymbol{h}_q(\partial\nabla s', \partial\nabla s'-\nabla^\# s),$$
$$\sigma(a)=\mu(a)^{-1}\nu(a),$$

where $\nabla^\#(\cdot)=\nabla(\cdot)-\partial\nabla(\cdot)$ and $\boldsymbol{h}_*=$any general homology theory, q any integer. For distinct $a_1, \cdots, a_k\in s\subset U$ $(k\geqq 1)$, denote $s-\{a_1, \cdots, a_k\}$ as s'' and define

(d, Cont'd.) $\quad\sigma(a_*):\boldsymbol{h}_q(\nabla s, \partial\nabla s)\xrightarrow{\cong}\boldsymbol{h}_{q+k}(\nabla s'', \partial\nabla s'')$

as $\sigma(a_k)\cdots\sigma(a_2)\sigma(a_1)$, where a_* means $(a_1, \cdots, a_n)$. $\sigma(a_*)$ is alternating, because, for any permutation of a_*, the corresponding coordinate transformation of $\nabla s''$ permutes the factors of $\sigma(a_*)$ in the same way.

Now let $a_*=(a_0, \cdots, a_{NU})$ be a choice of numbering of U, and for each nonempty subset $s\subset I$ let $i^s_*=(i^s_0, \cdots, i^s_{Ns})$ be a choice of numbering of s. They, together with σ and Lemmas 1, 2 below, determine two isomorphisms:

(e)

$$
\begin{tikzcd}
h_q\left(\bigcap_I X_i, \bigcap_I A_i\right) \arrow[d, "\cong"', "(-1)^{jNU}\sigma(a_*)"] \arrow[dddd, dashed, bend right=80, "\Psi_q"'] \\
h_{q+NU+1}\left(\left(\bigcap_I X_i, \bigcap_I A_i\right) \times (\nabla\varnothing, \partial\nabla\varnothing)\right) \arrow[d, "(\text{Lemma 1})\ \cong"', "\partial_{q+NU+1} \text{ of the homology theory } h_* = h_*((\bigcap_I X_i, \bigcap_I A_i)\times(\cdot))"] \arrow[ddd, dashed, bend right=60, "\lambda_\varnothing"'] \\
h_{q+NU}\left(\left(\bigcap_I X_i, \bigcap_I A_i\right) \times (K, C)\right) \arrow[d, "(\text{Lemma 1})\ \cong"', "h_{q+NU}(\subset)"] \\
h_{q+NU}(L, M) \arrow[d, equal] \\
G_q
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bigoplus_{Ns=n;\, s\subset I} h_j\left(\bigcup_s X_i, \bigcup_s A_i\right) \arrow[d, "\cong"', "(-1)^{jNU}\cdot(\bigoplus \sigma(i_*{}^s)^{-1}\sigma(a_*))"] \arrow[dddd, dashed, bend right=80, "\varphi_j^n"'] \\
\bigoplus_{Ns=n;\, s\subset I} h_p\left(\left(\bigcup_s X_i, \bigcup_s A_i\right) \times (\nabla s, \partial\nabla s)\right) \arrow[d, "(\text{Lemma 2})\ \cong"', "\Sigma h_p(\subset)"] \arrow[ddd, dashed, bend right=60, "\sum \lambda_s"'] \\
h_p(L^n, L^{n+1}) \arrow[d, equal] \\
E_{1;j}^n
\end{tikzcd}
$$

$(q, n, j \in \boldsymbol{Z})$ which combine with formulas (c) to give the formula (a).

Lemma 1. *$\lambda_\varnothing$ is an isomorphism.*

Proof. (Referring to (e).) ∂_{q+NU+1} is bijective by contractibility of $(\nabla\varnothing, C) = \bigwedge_I T \wedge (\bigwedge_{U-I} T, \partial \bigwedge_{U-I} T)$. For bijectivity of $h_{q+NU}(\subset)$ it suffices by the Five Lemma to consider the case $A_i = \varnothing$ (all $i \in I$). Define

$$L^{(n)} = X \times C \cup \bigcup_{s\subset I;\, Ns\geqq n} \left[\bigcap_s X_i \times \bigcup_s \nabla\{i\}\right] \qquad (n \in \boldsymbol{Z}).$$

$$h_*(L^{(n)}, L^{(n+1)}) \cong \bigoplus_{Ns=n} h_*\left(\left(\bigcap_s X_i, \bigcup_{I-s} X_i \cap \bigcap_s X_i\right) \times \left(C \cup \bigcup_s \nabla\{i\}, C\right)\right)$$

by additivity of homology, $\cong\{0\}$ by contractibility of

$$\Big(C \cup \bigcup_{s} \nabla\{i\}, C\Big) = \bigwedge_{I-s} T \wedge \Big(\partial \bigwedge_{s\cup(U-I)} T, \bigwedge_{s} T \wedge \partial \bigwedge_{U-I} T\Big),$$

assuming $0 \leqq n < NI$. $h_*(\bigcap_I X_i \times (K, C)) \cong h_*(L^{(NI)}, X \times C)$ by excision, $\cong h_*(L^{(NI-1)}, X \times C) \cong \cdots \cong h_*(L^{(0)}, X \times C) = h_*(L, X \times C)$ by exactness using above $\{0\}$. $\square$

LEMMA 2. *$\sum \lambda_s$ is an isomorphism.*

PROOF. Additivity of homology. $\square$

Comparison with the Mayer-Vietoris sequence. Since (a) relates the various unions of the pairs $\mathscr{P}$ to their intersection it brings to mind the Mayer-Vietoris sequence. (a) is in fact a generalization of the latter, as we shall now show. (The Mayer-Vietoris sequence is the $NI=1$ case of ε, δ^0, β below.)

For any $n \in \boldsymbol{Z}$ let $S_n(I) = \{i_* = (i_0, \cdots, i_n) \mid i_0, \cdots, i_n \in I\}$, which is to entail that $S_n(I) = \varnothing$ for negative n. Using X_{i_*} to mean $X_{i_0} \cup \cdots \cup X_{i_n}$, define $C^n(\mathscr{P}; h_j(\bigcup \cdot))$ $(j \in \boldsymbol{Z})$ = subgroup of $\prod_{i_* \in S_n(I)} h_j(X_{i_*}, A_{i_*})$ consisting of alternating members $\xi = \{\xi^{i_*} \mid i_* \in S_n(I)\}$ for which $\xi^{i_*} = 0$ whenever two or more of $i_0, \cdots, i_n$ are equal, and note that

$$C^n(\mathscr{P}; h_j(\bigcup \cdot)) \cong \bigoplus_{Ns=n;\, s\subset I} h_j\Big(\bigcup_s X_i, \bigcup_s A_i\Big)$$

under the correspondence $\xi \mapsto \{\xi^{i_*^s} \mid Ns = n, s \subset I\}$. Denote by Φ_j^n the composite of φ_j^n with this isomorphism.

LEMMA 3. *The following diagram commutes:*

$$\begin{tikzcd}
h_j\Big(\bigcap_I X_i, \bigcap_I A_i\Big) \arrow[rrr, "\varepsilon"] \arrow[d, "\Psi_j", "\cong"'] & & & C^0(\mathscr{P}; h_j(\bigcup \cdot)) \arrow[r, "\delta^0"] \arrow[d, "\Phi_j{}^0", "\cong"'] & C^1(\mathscr{P}; h_j(\bigcup \cdot)) \arrow[r, "\delta^1"] \arrow[d, "\Phi_j{}^1", "\cong"'] & \cdots \\
G_j \arrow[r, two heads, dashed] \arrow[rrr, bend left, "\kappa_j"] & F^0G_j/F^1G_j \arrow[r, dashed, "="] & E^0_{\infty;j} \arrow[r, tail, dashed] & E^0_{1;j} \arrow[r, "d_{1;j}"] & E^1_{1;j} \arrow[r, "d^1"] & \cdots
\end{tikzcd}$$

where

$$\varepsilon(\xi)^{i_0} = \xi\big|^{X_{i_0}, A_{i_0}} \qquad \Big(\xi \in h_j\Big(\bigcap_I X_i, \bigcap_I A_i\Big), i_0 \in I\Big)$$

and

$$\delta^n(\xi)^{i_*} = \sum_{0\leqq k\leqq n+1} (-1)^k \xi^{i_*(k)}\big|^{X_{i_*}, A_{i_*}}$$
$$(n \in \boldsymbol{Z},\ \xi \in C^n(\mathscr{P}; h_j(\bigcup \cdot)),\ i_* \in S_{n+1}(I))$$

$i_*(k)$ being $(i_0, \cdots, i_{k-1}, i_{k+1}, \cdots, i_{n+1})$.

PROOF. For the square involving δ^n for some n, $0\leqq n\leqq NI$, we consider an arbitrary element of $C^n(\mathscr{P}; h_j(\bigcup\cdot))$ of the form $\chi(i_*;\theta)$ defined as follows: i_* is a numbering of a subset $s\subset I$ with $Ns=n$, θ belongs to $h_j(\bigcup_s X_i, \bigcup_s A_i)$, and $\chi(i_*;\theta)^{i_*'}=\pm\theta$ or 0, depending upon whether i_*' is an even or odd permutation of i_* or not a permutation of i_*, respectively. $\delta^n\chi(i_*;\theta)=\sum_{i\in I-s}\chi(ii_*;\theta|^{X_{ii_*},\, A_{ii_*}})$. In the commutative diagram

$$\begin{tikzcd}
 & \boldsymbol{h}_p(\nabla s, \partial\nabla s) \arrow[rr, "\{\mu(i)\}"] \arrow[dl, "\lambda_s"'] \arrow[dr, "\partial_p"] & & \bigoplus \boldsymbol{h}_{p-1}(\partial\nabla s, \partial\nabla s - \nabla^{\#}(s\cup i)) & \bigoplus \boldsymbol{h}_{p-1}(\nabla(s\cup i), \partial\nabla(s\cup i)) \arrow[l, "\oplus\nu(i)"', "\cong"] \arrow[dll, "\cong"', "\sum \boldsymbol{h}_{p-1}(\subset)"] \arrow[dd, "\oplus R^{ii_*}_{i_*}"] \\
h_p(L^n, L^{n+1}) \arrow[dr, "d^n_{1;j}"'] & & \boldsymbol{h}_{p-1}(\partial\nabla s, \partial\nabla s - \cup\,\nabla^{\#}(s\cup i)) \arrow[ur, "\cong"', "\{\boldsymbol{h}_{p-1}(\subset)\}"] \arrow[dl, "\boldsymbol{h}_{p-1}(\subset)"'] & & \\
 & h_{p-1}(L^{n+1}, L^{n+2}) & & & \bigoplus h_{p-1}((X_{ii_*}, A_{ii_*})\times(\nabla(s\cup i), \partial\nabla(s\cup i))) \arrow[llll, "\sum\lambda_{s\cup i}"']
\end{tikzcd}$$

assume $\boldsymbol{h}_*=h_*((X_{i_*}, A_{i_*})\times(\cdot))$, that each $R^{ii_*}_{i_*}$ is induced by the appropriate inclusion, and each sum or union is taken over $\{i|i\in I-s\}$. We have that $d^n_{1;j}\lambda_s=h_{p-1}(\subset)\partial_p$, etc., $=\sum\lambda_{s\cup i}R^{ii_*}_{i_*}\sigma(i)^{-1}$. Therefore,

$$\begin{aligned}
(-1)^{jNU}d^n_{1;j}\Phi^n_j\chi(i_*;\theta) &= d^n_{1;j}\lambda_s\sigma(i_*)^{-1}\sigma(a_*)\theta \\
&= \sum \lambda_{s\cup i}R^{ii_*}_{i_*}\sigma(ii_*)^{-1}\sigma(a_*)\theta \\
&= \sum \lambda_{s\cup i}\sigma(ii_*)^{-1}\sigma(a_*)(\theta|^{X_{ii_*},A_{ii_*}}) \\
&= (-1)^{jNU}\cdot\Phi^{n+1}_j\delta^n\chi(i_*;\theta).
\end{aligned}$$

We have thus proved the square commutative, since the $\chi(i_*;\theta)$'s generate $C^n(\mathscr{P}; h_j(\bigcup\cdot))$. For the square involving ε, the same argument works with $\boldsymbol{h}_*$ redefined as $h_*((\bigcap_I X_i, \bigcap_I A_i)\times(\cdot))$, s replaced by $\varnothing$, and $d^{\,n}_{1;j}$ replaced by κ_q. $\square$

LEMMA 4. *If $A_i=\varnothing$ (all $i\in I$), the following diagram commutes:*

$$\begin{tikzcd}
C^{NI}(\mathscr{P}; h_j(\bigcup\cdot)) \arrow[rrr, "\beta"] \arrow[d, "\cong"', "\Phi^{NI}_j"] & & & h_{j-NI}\left(\bigcap_I X_i\right) \arrow[d, "\cong"', "\Psi_{j-NI}"] \\
E^{NI}_{1;j} \arrow[r, two heads] \arrow[rrr, bend right] & E^{NI}_{\infty;j} \arrow[r, "="] & F^{NI}G_{j-NI}/F^{NI+1}G_{j-NI} \arrow[r, tail] & G_{j-NI}
\end{tikzcd}$$

where β has the formula

$$C^{NI}(\mathscr{P};\, h_j(\bigcup\,\cdot\,)) \cong h_j\Big(\bigcup_I X_i\Big) \to h_j\Big(\bigcup_I X_i,\, \bigcup_{j<NI} X_{(j)}\Big)$$

$$\xrightarrow[\text{excision}]{\cong} h_j\Big(X_{(NI)},\, \bigcup_{j<NI} X_{(j)} \cap X_{(NI)}\Big)$$

$$\xrightarrow{(-1)^{(NI)^2}\beta_1\beta_2\cdots\beta_{NI}} h_{j-NI}\Big(\bigcap_I X_i\Big) \qquad (X_{(j)} = X_{i_jI}),$$

each β_k $(1 \leqq k \leqq NI)$ being the composite

$$\begin{array}{c} h_{j-NI+k}\Big(\bigcap_{j\geqq k} X_{(j)},\, \bigcup_{j<k} X_{(j)} \cap \bigcap_{j\geqq k} X_{(j)}\Big) \\ \big\downarrow \partial_{j-NI+k} \\ h_{j-NI+k-1}\Big(\bigcup_{j<k} X_{(j)} \cap \bigcap_{j\geqq k} X_{(j)},\, \bigcup_{j<k-1} X_{(j)} \cap \bigcap_{j\geqq k} X_{(j)}\Big) \\ \cong \big\downarrow \text{excision} \\ h_{j-NI+k-1}\Big(\bigcup_{j\geqq k-1} X_{(j)},\, \bigcup_{j<k-1} X_{(j)} \cap \bigcap_{j\geqq k-1} X_{(j)}\Big). \end{array}$$

PROOF. Omitted. Consists of comparing each β_k with the appropriate form of $\sigma(i_j^I)^{-1}$ in one large commutative diagram. $\square$

Independence from U. Let $U^+ = U \oplus \{a\}$ for some point a apart from U, and indicate by a superscript $^+$ the U^+-version of each of the notions (b)–(e). To prove that the choice of U is immaterial it suffices to prove (c)$\cong$(c$^+$), (e)$\cong$(e$^+$). We therefore define an isomorphism

$$l^{n,m}: h_p(L^n, L^m) \to h_{p^+}(L^{+n}, L^{+m})$$

as follows, for n, j as in (c) and $m \geqq n$:

$$h_p(L^n, L^m) = \boldsymbol{h}_p^{(n,m)}\Big(\partial \bigwedge_{\{a\}} T,\, \{*\}\Big)$$

$$\xrightarrow[\cong]{\partial_{p+1}^{-1}} \boldsymbol{h}_{p+1}^{(n,m)}\Big(\bigwedge_{\{a\}} T,\, \partial \bigwedge_{\{a\}} T\Big) \xrightarrow[\text{excision}]{\cong} h_{p^+}(L^{+n}, L^{+m}).$$

Here $\boldsymbol{h}_*^{(n,m)}$ is the general homology theory of *based* compact pairs (Y, B) with formula $\boldsymbol{h}_*^{(n,m)}(Y, B) = h_*(L^n(Y), L^m(Y) \cup L^n(B))$, $L^n(Y)$ being $X \times C \wedge Y \cup \bigcup_{s\subset I} [\bigcup_s A_i \times \nabla s \wedge Y] \cup \bigcup_{s\subset I; Ns\geqq n} [\bigcup_s X_i \times \nabla s \wedge Y]$. Then, $(-1)^{j-n-1} l^{n,n+r}$ induces an isomorphism $E_{r;j}^n \to E_{r;j}^{+n}$ $(n, j \in \boldsymbol{Z};\, r = 1, 2, \cdots)$ that carries $d_{r;j}^n$ into $d_{r;j}^{+n}$, Φ_j^n (for $r=1$) into Φ_j^{+n}, etc., as required. We assume that $a_*^+ = a_* a$.

Functoriality. Constructing (a) is more difficult than constructing the spectral sequence of a cover in that the underlying spaces (b) do not depend functorially on $(X, I, \mathscr{P})$. U has been introduced as a remedy.

We assume that a *morphism* from $(X, I, \mathscr{P})$ to another such triple $(Y, J, \mathscr{Q})$, $\mathscr{Q}$ being a finite family $\{(Y_j, B_j) \mid j \in J\}$ of subspace pairs in a space Y, is a map $f: X \to Y$ of spaces together with a map $\pi: J \to I$ of sets such that $(fX_{\pi j}, fA_{\pi j}) \subset (Y_j, B_j)$ $(j \in J)$. Evidently $C^n(\mathscr{P}; h_q(\bigcup \cdot))$ $(n, q \in \boldsymbol{Z})$ depends functorially on $(X, I, \mathscr{P})$ if $(f; \pi)$ is regarded as inducing the map $C^n(f; \pi): C^n(\mathscr{P}; h_q(\bigcup \cdot)) \to C^n(\mathscr{Q}; h_q(\bigcup \cdot))$ with the formula $(C^n(f; \pi)\xi)^{j_*} = h_q(f; \pi)^{j_*} \xi^{\pi j_*}$ $(\xi \in C^n(\mathscr{P}; h_q(\bigcup \cdot)), j_* \in S_n(J))$, $h_q(f; \pi)^{j_*}$ being the homomorphism $h_q(X_{\pi j_*}, A_{\pi j_*}) \to h_q(Y_{j_*}, B_{j_*})$ induced by $f|_{X_{\pi j_*}}$. Similarly, $h_*(\bigcap_I X_i, \bigcap_I A_i)$ is functorial, the induced map to be denoted $h_*(f; \bigcap)$.

Let primes signify the $(Y, J, \mathscr{Q})$-version of the notions (b)–(e). To show that (c), Ψ_*, Φ_*^* depend functorially on $(X, I, \mathscr{P})$, we need only produce a homomorphism of (c) to (c′) which, when considered along with $C^*(f; \pi)$ and $h_*(f; \bigcap)$, maps Ψ_*, Φ_*^* to Ψ'_*, Φ'^*_* respectively. It is easy to see that this map of (c) to (c′) is *a fortiori* unique and functorially dependent on the morphism $(f; \pi)$.

We start by assuming $U' = U \supset I \oplus J$. Define $\omega: \bigwedge_U T \to \bigwedge_U T$ to be the involution $\bigwedge_{U-(\pi J \cup J)} 1_T \wedge \bigwedge_{\pi J} \omega_i$, where, for each $i \in \pi J$,

$$\omega_i\left(t_i \wedge \bigwedge_{\pi^{-1}\{i\}} t_j\right) = m \wedge \bigwedge_{\pi^{-1}\{i\}} (t_j t_i / m)$$

$(t_i, t_j \in T$ for $j \in \pi^{-1}\{i\})$, m being $\operatorname{Max}_{\pi^{-1}\{i\}} t_j$. It is easily shown that $\omega \nabla\{i\} = \bigcup_{\pi^{-1}\{i\}} \nabla\{j\}$ for $i \in \pi J$, while $\omega \nabla\{i\} \subset C'$ for $i \in I - \pi J$. The consequence is $(f \times \omega) L^n \subset L'^n$ $(n \in \boldsymbol{Z})$, with an induced homomorphism $l^{n,m}: h_p(L^n, L^m) \to h_p(L'^n, L'^m)$ $(m \geqq n)$. The map $(-1)^{\text{number of members in } \pi J} \cdot l^{n,n+r}$ induces the required $E^n_{r;j} \to E'^n_{r;j}$ $(n, j \in \boldsymbol{Z}; r = 1, 2, \cdots)$. (The power of (-1) is the degree of ω.)

The homology sheaf. Let $\mathscr{P} = \{(X, A \cup (X - U^i)) \mid i \in I\} = \mathscr{P}_{\mathscr{U}}$ for some finite open cover $\mathscr{U} = \{U^i \mid i \in I\}$ of X, A being some subspace. Evidently

$$h_*\left(\bigcap_I X_i, \bigcap_I A_i\right) = h_*(X, A),$$

$$C^*(\mathscr{P}_{\mathscr{U}}; h_*(\bigcup \cdot)) = C^*(\mathscr{U}; h_*^{X,A}),$$

where $h_*^{X,A}$ is the graded presheaf $\{h_*(X, A \cup (X - \mathscr{O})) \mid \text{open } \mathscr{O} \subset X\}$. Thus, we obtain a spectral sequence

$$\text{(f)} \qquad E^n_{2;j} \cong H^n(\mathscr{U}; h_j^{X,A}) \underset{n}{\Rightarrow} h_{j-n}(X, A).$$

For X compact, the direct limit of (f), as $\mathcal{U}$ is refined, is a spectral sequence

$$E^n_{2;j} \cong H^n(X; \mathscr{h}^{X,A}_j) \underset{n}{\Rightarrow} h_{j-n}(X, A), \tag{g}$$

where $\mathscr{h}^{X,A}_j$ is the induced sheaf of $h^{X,A}_j$. As A approximates an open set V from within, the direct limit of (g) is

$$E^n_{2;j} \cong H^n(X, V; \mathscr{h}^X_j) \underset{n}{\Rightarrow} h_{j-n}(X, V). \tag{h}$$

$\mathscr{h}^X_*$ is called the homology sheaf of X. If $\mathscr{h}^X_j \cong \{0\}$ except for $j=j_0$ ($=$ some integer), e.g., if X is a j_0-manifold and h_* is standard, then (h) collapses to a family of isomorphisms

$$H^n(X, V; \mathscr{h}^X_{j_0}) \cong h_{j_0-n}(X, V) \qquad (n \in \boldsymbol{Z}).$$

(Compare to [**2**].)

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