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FORCING AND MODELS OF ARITHMETIC[1]

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ABSTRACT. It is shown that every countable model of Peano arithmetic with finitely many extra predicates (or of ZFC with finitely many extra predicates) is a reduct of a pointwise definable such model.

This note applies the forcing method to a question concerning definability in models of Peano arithmetic.

THEOREM. *Let* $M=\langle |M|, +, \cdot\rangle$ *be a countable model of Peano arithmetic.*[2] *Then there is a set* $U\subseteq |M|$ *such that*

(i) $\langle M, U\rangle$ *satisfies the first order induction schema for formulas containing an extra predicate* $U(x)$;

(ii) *every element of* $|M|$ *is first order definable in* $\langle M, U\rangle$.

PROOF. A *condition* is an M-finite sequence of 0's and 1's, i.e. a mapping $p:\{b|b<^M a\}\to\{0, 1\}$ such that $a\in |M|$ and p is coded by an element of $|M|$. We use $p, q, \cdots$ as variables ranging over conditions. A set of conditions is *dense* if every condition is extended by some condition in the set. Let $\langle a_n|n<\omega\rangle$ enumerate the elements of $|M|$. Let $\langle D_n|n<\omega\rangle$ enumerate the dense sets of conditions which are first order definable over M allowing parameters from $|M|$. It is safe to assume:

(*) the parameters in the first order definition of D_n are among $a_0, a_1, \cdots, a_{n-1}$.

Define a sequence of conditions $\langle p_n|n<\omega\rangle$ by $p_0=\varnothing$; $p_{2n+1}=$the $<^M$-least condition $q\supseteq p_{2n}$ such that $q\in D_n$; $p_{2n+2}=p_{2n+1}$ followed by a string of a_n 0's followed by a 1. Define $U\subseteq |M|$ by letting $\bigcup\{p_n|n<\omega\}$ be the characteristic function of U. To prove that $\langle M, U\rangle$ satisfies first order induction, use the genericity of U.

[*Details.* Let L be the first order language with $+, \cdot, U(x)$, and constant symbols $\boldsymbol{a}$ for each $a\in |M|$. For θ a sentence of L define the (strong)

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[2] Peano arithmetic is the theory P of Shoenfield, *Mathematical logic*, Addison-Wesley, 1967.

forcing relation $p \Vdash \theta$ by

$$
\begin{array}{lll}
p \Vdash \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} & \text{iff} & a + b = c; \\
p \Vdash \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{c} & \text{iff} & a \cdot b = c; \\
p \Vdash \boldsymbol{U}(\boldsymbol{a}) & \text{iff} & p(a) = 1; \\
p \Vdash \theta_1 \vee \theta_2 & \text{iff} & p \Vdash \theta_1 \text{ or } p \Vdash \theta_2; \\
p \Vdash \neg\theta & \text{iff} & q \Vdash \theta \text{ for no } q \supseteq p; \\
p \Vdash \exists x\theta(x) & \text{iff} & p \Vdash \theta(\boldsymbol{a}) \text{ for some } a \in |M|.
\end{array}
$$

Prove the basic forcing lemmas as usual. It remains to show that $\varnothing \Vdash \neg\neg(\exists x\theta(x) \rightarrow \exists$ least x such that $\theta(x))$.

Suppose $p \Vdash \exists x\theta(x)$. Then $p \Vdash \theta(\boldsymbol{a})$ for some $a \in |M|$. Working within M, define a sequence of conditions $\langle q_c | c <^M b+1 \rangle$ where $b <^M a+1$ as follows: $q_0 = p$; q_{c+1}=the $<^M$-least $q \supseteq q_c$ such that $q \Vdash \neg\theta(\boldsymbol{c})$; b=the $<^M$-least c such that q_{c+1} is undefined. Then $q_b \supseteq p$ and $q_b \Vdash \neg\neg(\boldsymbol{b}$ is the least x such that $\theta(x))$.]

On the other hand, using (∗), one easily shows by induction on n that p_{2n+1}, p_{2n+2}, and a_n are first-order definable in $\langle M, U \rangle$. This completes the proof.

Remark. One can apply the same method to models of set theory to get the following theorem: *Let $M = \langle |M|, \in^M \rangle$ be a countable model of ZFC, then there is a set $U \subseteq |M|$ such that*

(i) *$\langle M, U \rangle$ satisfies the first-order replacement schema for formulas containing an extra predicate $\boldsymbol{U}(x)$;*

(ii) *every element of $|M|$ is first-order definable in $\langle M, U \rangle$.*

This is an improvement of the theorem of U. Felgner, Fund. Math. **71** (1971), 43–62.

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