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A NOTE ON DEDEKIND RINGS

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ABSTRACT. We prove the following theorem: *let R be a commutative ring without zero-divisors. If every ideal in R is a product of prime ideals* $\neq R$, *R has the identity.*

To prove the theorem stated in the abstract, let R be a ring satisfying the hypotheses and K its quotient field. We denote $(R:R)_K$ by R^{-1}. The facts $R^{-1} \ni 1$ and $R^{-1}R=R$ are immediate by the definition. If aR^{-1} coincides with R for all $a\neq 0$ of R, we have $R=a^2R^{-1}=aR$.

Then we have $a=ab$ for some b of R. We see that b is the identity of R. We may suppose therefore that dR^{-1} is a proper subset of R. Since dR^{-1} is an ideal, there exist prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ such that

$$dR^{-1} = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_n \qquad (n \geqq 1).$$

There exist prime ideals $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_m$ also such that $\mathfrak{p}_nR=\mathfrak{g}_1\mathfrak{g}_2\cdots\mathfrak{g}_m$. Every $\mathfrak{g}_i$ contains $\mathfrak{p}_n$ and $\mathfrak{p}_n$ contains $\mathfrak{g}_j$ for some j, say 1. If $\mathfrak{p}_nR$ is a proper subset of $\mathfrak{p}_n$, we have

$$\mathfrak{p}_nR = \mathfrak{p}_n\mathfrak{g}_2 \cdots \mathfrak{g}_m \qquad (m \geqq 2).$$

Multiplying $d^{-1}R \cdot \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_{n-1}$ on the both sides, we have $R^2=R\mathfrak{g}_2 \cdots \mathfrak{g}_m$. There arises the contradiction of $R\subset \mathfrak{g}_m$. Therefore $\mathfrak{p}_nR$ coincides with $\mathfrak{p}_n$. We have

$$dR^{-1}R = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_nR = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_n = dR^{-1}.$$

Therefore we see that $R^{-1}R$ is R^{-1}, hence we have

$$R = R^{-1}R = R^{-1} \ni 1.$$

This concludes the proof of our assertion.

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