CLASS NUMBERS AND $\mu$-INVARIANTS OF CYCLOTOMIC FIELDS

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ABSTRACT. We give a new upper bound for the $\mu$-invariant of a cyclotomic field by estimating the first factor of the class number of the $p$th cyclotomic field ($p$ an odd prime).

For each $n \geq 0$ let $h_n$ denote the class number of the cyclotomic field of $p^{n+1}$th roots of unity, where $p$ is an odd prime. According to Iwasawa [1], the greatest exponent $e(n)$ for which $p^{e(n)} | h_n$ is given by a formula

$$e(n) = \lambda n + \mu p^n + \nu,$$

valid for all sufficiently large $n$. Here $\lambda$, $\mu$, and $\nu$ are integers ($\lambda, \mu \geq 0$) independent of $n$. In [2] Iwasawa proved the following estimates for $\mu$:

(i) $\mu < p - 1$ for all $p$,
(ii) if $c > \frac{1}{2}$, then there exists a bound $N(c)$ such that $\mu < c(p - 1)$ whenever $p > N(c)$.

We shall show that $\mu < (p - 1)/2$ for all $p$.

Let us denote by $h^\prime$ the so-called first factor of $h_0$. As shown in [2], the problem of estimating $\mu$ can be reduced to that of estimating $h^\prime$ by means of the relation $p^{\mu/2} \leq h^\prime$.

It is known that

$$h^\prime = (2p)^{-(p-3)/2} \left| \prod_{\chi \in S} \sum_{n=1}^{p-1} \chi(n)n \right|,$$

where $S$ denotes the set of all odd residue class characters mod $p$. Noting that

$$\sum_{\chi \in S} \chi(m)\chi'(n) = (p - 1)/2 \quad \text{if } m \equiv n \pmod{p}, (mn, p) = 1,$$

$$= -(p - 1)/2 \quad \text{if } m \equiv -n \pmod{p}, (mn, p) = 1,$$

$$= 0 \quad \text{otherwise},$$

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\( (\chi' \text{ means the complex conjugate of } \chi), \text{ we first get} \)
\[
\sum_{\chi \in \mathcal{S}} \left| \sum_{n=1}^{p-1} \chi(n)n \right|^2 = \left( \frac{(p-1)/2}{\sum_{n=1}^{p-1} n^2 - \sum_{n=1}^{p-1} n(p-n)} \right) = (p-2)(p-1)^2 p/12.
\]

Therefore, by the arithmetic-geometric mean inequality,
\[
\prod_{\chi \in \mathcal{S}} \left| \sum_{n=1}^{p-1} \chi(n)n \right|^{4/(p-1)} \leq (p-2)(p-1)p/6 < p^3/6.
\]

This gives us the estimate

\[
(1) \quad h^- < 2p(p/24)^{(p-1)/4}
\]

Thus, if \( p > 3 \), we see that \( h^- < p^{(p-1)/4} \) and so \( \mu < (p-1)/2 \). This holds also for \( p = 3 \), since then \( h^- = 1 \). (As a matter of fact, we know that \( \mu = 0 \) for all regular primes.)

It should be mentioned that the result (1) has been obtained earlier by Lepistö [3] and the author [4] by more complicated methods than that presented above.

**References**


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