STARLIKE FUNCTIONS

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Abstract. Let \( \mathcal{S}^*[\alpha] \) denote the class of functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) analytic in \( |z| < 1 \) and for which \( |zf'(z)/f(z) - 1| < 1 - \alpha \) for \( |z| < 1 \) and \( \alpha \in [0, 1) \). Sharp results concerning coefficients, distortion, and the radius of convexity are obtained. Furthermore, it is shown that \( \sum_{n=2}^{\infty} [(n-\alpha)/(1-\alpha)] |a_n| < 1 \) is a sufficient condition for \( f(z) \in \mathcal{S}^*[\alpha] \).

1. Introduction. Suppose that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic and \( \Re \{zf'(z)/f(z)\} > \alpha \) for \( |z| < 1 \) and \( \alpha \in [0, 1) \), then \( f(z) \) is called starlike of order \( \alpha \), denoted by \( f(z) \in \mathcal{S}^*_\alpha \). It was shown by Schild [7] that for \( f(z) \in \mathcal{S}^*_\alpha \) the domain of values of \( \{zf'(z)/f(z)\} \) is the circle with line segment from \( (1 + (2\alpha - 1)|z|)/(1 + |z|) \) to \( (1 - (2\alpha - 1)|z|)/(1 - |z|) \) as a diameter. In this paper we consider a subclass of \( \mathcal{S}^*_\alpha \) consisting of those \( f(z) \) for which \( |zf'(z)/f(z) - 1| < 1 - \alpha \) for \( |z| < 1 \) and denote it by \( \mathcal{S}^*[\alpha] \). Sharp results concerning coefficients, distortion, and the radius of convexity are obtained. Furthermore, it is shown that \( \sum_{n=2}^{\infty} [(n-\alpha)/(1-\alpha)] |a_n| \leq 1 \) is a sufficient condition for \( f(z) \) to be in \( \mathcal{S}^*[\alpha] \). Results about \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \) have previously been the subject of papers by Goodman [3], MacGregor [4], and Schild [6].

2. Coefficient theorems.

Theorem 1. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and suppose that
\[
\sum_{n=2}^{\infty} [(n-\alpha)/(1-\alpha)] |a_n| \leq 1;
\]
then \( f(z) \in \mathcal{S}^*[\alpha] \) for \( \alpha \in [0, 1) \).

Proof. Let \( |z| = 1 \), then
\[
|zf'(z) - f(z)| - (1 - \alpha) |f(z)|
= \left| \sum_{n=2}^{\infty} (n-1)a_n z^n \right| - (1 - \alpha) \left| z + \sum_{n=2}^{\infty} a_n z^n \right|
\leq \sum_{n=2}^{\infty} (n-\alpha) |a_n| - (1 - \alpha) \leq 0
\]

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by the hypothesis. Hence \(|zf'(z)/f(z) - 1| < 1 - \alpha\) for \(|z| < 1\) by the maximum modulus theorem. We note that \(f(z) = z - [(1 - \alpha)/(n - \alpha)]z^n\) is an extremal function with respect to the theorem since \(|zf'(z)/f(z) - 1| = 1 - \alpha\) for \(z = 1, \alpha \in [0, 1]\), and \(n = 2, 3, \cdots\). We also observe that the converse to

\[
\sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} |a_n| = \sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} \frac{(1 - \alpha)^{n-1}}{(n-1)!} > 2e^{1-\alpha} - 1 > 1 \quad \text{for all } \alpha \in [0, 1).
\]

For \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_a^\star\) it has been shown [7] that the sharp inequality \(|a_n| \leq \prod_{k=2}^{n-2} (k - 2\alpha)/(n - 1)!\) holds for \(n = 2, 3, \cdots\) and is attained by \(f(z) = z(1 - z)^{-2(1 - \alpha)}\). It is interesting to note that \((1 - \alpha)/(n - 1)\) appears as a factor of the upper bound for each \(|a_n|\). For our class it is the upper bound.

**Theorem 2.** Let \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_a^\star\), then \(|a_n| \leq (1 - \alpha)/(n - 1)\) for \(n = 2, 3, \cdots\) and \(\alpha \in [0, 1)\).

**Proof.** If \(f(z) \in \mathcal{S}_a^\star\), then \(|zf'(z)/f(z) - 1| < 1 - \alpha\) and since the absolute value vanishes for \(z = 0\) we obtain

\[
zf'(z)/f(z) = 1 + \omega(z)
\]

where \(\omega(z) = \sum_{k=1}^{\infty} \omega_k z^k\) is analytic and \(|\omega(z)| < 1 - \alpha\) for \(|z| < 1\). From (1) we see that \(zf'(z) - f(z) = f(z)\omega(z)\) or equivalently

\[
\sum_{k=2}^{\infty} (k - 1)a_k z^k = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)\left(\sum_{k=1}^{\infty} \omega_k z^k\right).
\]

Equating coefficients on both sides of (2) shows that

\[n - 1)a_n = \omega_{n-1} + \sum_{k=2}^{n-1} a_1 \omega_{n-k}\]

for \(n = 2, 3, \cdots\), which implies that the coefficient \(a_n\) on the left side of (2) is dependent only on \(a_2, a_3, \cdots, a_{n-1}\) on the right side of (2). Hence for \(n \geq 2\)

\[
\sum_{k=2}^{n} (k - 1)a_k z^k + \sum_{k=n+1}^{\infty} A_k z^k = \left(z + \sum_{k=2}^{n-1} a_k z^k\right)\omega(z)
\]

for a proper choice of \(A_k\). Squaring the moduli of both sides of (3) and integrating around \(|z| = r < 1\) we get, using the fact that \(|\omega(z)| < 1 - \alpha\) for \(|z| < 1\),

\[
\sum_{k=2}^{n} (k - 1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |A_k|^2 r^{2k} < \left(1 + \sum_{k=2}^{n-1} |a_k|^2\right)(1 - \alpha)^2.
\]
Let \( r \to 1 \) and we find that
\[
\sum_{k=2}^{n} (k - 1)^2 |a_k|^2 \leq \left( 1 + \sum_{k=2}^{n-1} |a_k|^2 \right) (1 - \alpha)^2
\]
or
\[
(n - 1)^2 |a_n|^2 \leq (1 - \alpha)^2 - \sum_{k=2}^{n-1} [(k - 1)^2 - (1 - \alpha)^2] |a_k|^2 \leq (1 - \alpha)^2
\]
and it follows that \(|a_n| \leq (1 - \alpha)/(n - 1)\). This proof is based on a technique found in Clunie and Keogh [2]. For sharpness, consider the function
\[
f(z) = z \exp\left[\frac{(1 - \alpha)}{(n - 1)} z^{n-1}\right] = z + \left[\frac{(1 - \alpha)}{(n - 1)}\right] z^n + \cdots.
\]

3. Distortion theorems. In order to obtain distortion properties of \( f(z) \) and \( f'(z) \) we need the representation given by the following lemma.

**Lemma 1.** \( f(z) \in \mathcal{S}^*[\alpha] \) if and only if
\[
f(z) = z \cdot \exp\left(\int_{0}^{z} \phi(t) \, dt\right)
\]
where \( \phi(z) \) is analytic for \( |z| < 1 \) and \( |\phi(z)| \leq 1 - \alpha \) for \( |z| < 1 \) and \( \alpha \in [0, 1) \).

**Proof.** The “only if” is easily obtained by integrating (1) with \( \omega(z) = z\phi(z) \) while \( f(z) \in \mathcal{S}^*[\alpha] \) follows from differentiation and simple manipulation.

**Theorem 3.** If \( f(z) \in \mathcal{S}^*[\alpha] \), then
(i) \( |z| \, e^{(\alpha - 1) |z|} \leq |f(z)| \leq |z| \, e^{(1 - \alpha) |z|} \),
and
(ii) \( (1 + (\alpha - 1) |z|) e^{(\alpha - 1) |z|} \leq |f'(z)| \leq (1 + (1 - \alpha) |z|) e^{(1 - \alpha) |z|} \).

**Proof.** Since
\[
\left| \int_{0}^{z} \phi(t) \, dt \right| \leq \int_{0}^{|z|} |\phi(t)| \, dt \leq \int_{0}^{|z|} (1 - \alpha) \, dt \leq (1 - \alpha) |z|,
\]
then (i) follows by virtue of Lemma 1, and (ii) by applying the triangle inequality to \( f'(z) = (1 + z\phi(z)) \cdot f(z)/z \). Both parts of the theorem are sharp for \( f(z) = ze^{(1 - \alpha)z} \).

4. The radius of convexity. It is well known that every univalent function maps \( |z| < 2 - \sqrt{3} = 0.267 \cdots \) onto a convex region and that for the class of starlike functions \( \mathcal{S}^* \), this estimate cannot be improved since the extremal function for the class of univalent functions, the Koebe function \( K(z) = z/(1 - e^{i\theta}z)^2 \), is also starlike. Here we determine the exact
radius of convexity of $\mathcal{S}^*[\alpha]$ as a function of $\alpha$ for each $\alpha \in [0, 1)$. In particular, it is shown that for $\mathcal{S}^*[0]$ the radius of convexity is $(3-\sqrt{5})/2=0.382\cdots$.

**Theorem 4.** Suppose that $f(z) \in \mathcal{S}^*[\alpha]$; then $f(z)$ maps $|z|<r$ onto a convex domain for

(i) \[ r = (3 - \sqrt{5})/(2 - 2\alpha), \quad \text{if } \alpha \in [0, \alpha_0], \]

and

(ii) \[ r = \left[\left((-2\alpha^2 + \alpha - 1) + 2\alpha(6 - 6\alpha + \alpha^2)^{1/2}\right)/(1 + 3\alpha - 4\alpha^2)\right]^{1/2}, \quad \text{if } \alpha \in [\alpha_0, 1), \]

where $\alpha_0=1-(1+\sqrt{6})(3\sqrt{5}-5)/10=0.411\cdots$. The result is sharp.

**Proof.** As a notational convenience let $\beta = 1 - \alpha$. In [1] it is shown that if $\phi(z)$ is analytic for $|z|<1$ with $|\phi(z)| \leq 1$, then

\[ |\phi'(z)| \leq (1 - |\phi(z)|^2)/(1 - |z|^2). \]

If, however, $|\phi(z)| \leq \beta$ then we may apply the previous result to $\phi(z)/\beta$ and obtain

\[ |\phi'(z)| \leq (\beta - |\phi(z)|^\beta \beta^{-1})/(1 - |z|^\beta). \]

Let $\omega(z)=-z\phi(z)$ in (1); then after taking the logarithmic derivative of both sides we have

\[ \text{Re}\left(\frac{zf''(z)}{f'(z) + 1}\right) = \text{Re}\left(\frac{z^2\phi'(z) + z\phi(z)}{1 - z\phi(z)}\right). \]

Regrouping and then applying (4) and the triangle inequality to the right side of (5) gives us

\[ \text{Re}\{zf''(z)/f'(z) + 1\} \geq 1 - (|z\phi(z)| (1 + (1 - |z\phi(z)|)^{-1}) \]

\[ + |z|^2 |\phi'(z)| (1 - |z\phi(z)|)^{-1} \]

\[ \geq 1 - (|z\phi(z)| (2 - |z\phi(z)|)(1 - |z|^2) \]

\[ + |z|^2 (\beta - |\phi(z)|^\beta \beta^{-1})) \]

\[ \cdot ((1 - |z\phi(z)|)(1 - |z|^2))^{-1}. \]

For $f(z)$ to be convex in $|z|<r$ it suffices to have $\text{Re}\{zf''(z)/f'(z) + 1\} > 0$ in $|z|<r$, see [5]. In our case, (5) will certainly be positive when the right side of (6) is positive which is whenever

\[ \beta \cdot |z\phi(z)| (2 - |z\phi(z)|)(1 - |z|^2) + |z|^2 (\beta^2 - |\phi(z)|^\beta) \]

\[ < \beta(1 - |z\phi(z)|)(1 - |z|^2). \]
In order to discuss (7) let us consider the function
\[ p(x) = [(1 - r^2)\beta + 1)x^2 - 3(1 - r^2)\beta x - (\beta^2 r^2 - \beta(1 - r^2)) \]
where \( r = |z| \in [0, 1) \) and \( x = |z\psi(z)| \in [0, r\beta] \). Clearly
\[ p'(x) = 2[(1 - r^2)\beta + 1)x - 3(1 - r^2)\beta = 0 \]
for \( x = x_0 = 3(1 - r^2)/(2(1 - r^2) + 1) \) and \( p''(x) > 0 \). Hence, for each \( \beta \), \( p(x) \) is a parabola opening upward. Thus an investigation of (7) now reduces to trying to determine some relationship among \( \beta, r, \) and \( x \) so that \( p(x) > 0 \). In order to do so we explore the geometric significance of the two cases: (a) \( p'(r\beta) \leq 0 \) and (b) \( p'(r\beta) \geq 0 \).

(a) If \( p'(r\beta) \leq 0 \), then \( r\beta \leq x_0 \) and so \( p(x) \leq p(r\beta) \) for \( x \in [0, r\beta] \). Now, let \( p(r\beta) \) be considered as a function of \( r \) with \( \beta \) held constant, then (7) will be satisfied when
\[ p(r\beta) = \beta(1 - r^2)(1 - 3\beta r + \beta^2 r^2) > 0 \]
which is exactly when \( r < (3 - \sqrt{5})/(2\beta) \). This result, however, is valid only for those \( \beta \) for which \( r\beta \leq x_0 \) or equivalently, for \( \beta \) satisfying
\[ 2\sqrt{5}\beta^2 + (2\sqrt{5} - 6)\beta + (15 - 7\sqrt{5}) \geq 0. \]
Hence for \( \beta \in [\beta_0, 1] \) where \( \beta_0 = (1 + \sqrt{6})(3\sqrt{5} - 5)/10 \). The result is proved sharp by choosing the function \( f(z) = ze^{\beta z} \) for \( \beta \in [\beta_0, 1] \).

(b) On the other hand, \( p'(r\beta) \geq 0 \) implies \( r\beta \geq x_0 \) and \( p(x) \geq p(x_0) \) for \( x \in [0, r\beta] \). Again let \( \beta \) be fixed. Now (7) will be satisfied when
\[ p(x_0) = -\beta((5\beta - 4\beta^2)r^4 + 2(2\beta^2 - 3\beta + 2)r^2 + (5\beta - 4)) \cdot (4((1 - r^2)\beta + 1))^{-2} > 0 \]
which is whenever
\[ r < r_0 = [(-(2\beta^2 - 3\beta + 2) + 2(1 - \beta)(\beta^2 + 4\beta + 1)^{1/2})(5\beta - 4\beta^2)]^{1/2} \]
promised that \( r\beta \geq x_0 \), that is, for \( \beta \in (0, \beta_0] \).

We conclude with an existence proof for a sharp function. Suppose that \( \psi(z) = \beta(z - z_0)/(1 - z_0z) \) where \( z_0 \) is real; then
\[ z_0 = (\psi(z) - \beta z)/(z\psi(z) - \beta) \]
and so
\[ \psi'(z) = \beta(1 - z_0^2)/(1 - z_0z)^2 = (\beta^2 - \psi^2(z))/(\beta(1 - z^2)). \]
If we now let \( z\psi'(z) + 1 = -z\psi(z) \), then
\[ z\psi''(z) + 1 = (((1 - z^2)\beta + 1)(z\psi(z))^2 - 3(1 - z^2)\beta(z\psi(z)) - (\beta^2 z^2 - \beta(1 - z^2)))/((\beta(1 - z\psi(z))(1 - z^2)) = 0 \]
when \( z=r_0 \) and \( z\psi(z)=r_0\psi(r_0)=x_0 \). Since \( x_0 \leq r_0\beta \) we have

\[
 r_0\psi(r_0) = r_0\beta(r_0 - z_0)/(1 - r_0z_0) \leq r_0\beta
\]

and so \((r_0 - z_0)/(1 - r_0z_0) \leq 1\) which implies that \(|z_0| \leq 1\). Hence \(|\psi(z)| \leq \beta\)
for \(|z| < 1\) and so by Lemma 1

\[
f(z) = z \cdot \exp \left( \int_0^z (-\beta(t - z_0)/(1 - z_0t)) \, dt \right) \in \mathcal{S}^\#[1 - \beta].
\]

References

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