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ON PERIODICITY OF ENTIRE FUNCTIONS

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ABSTRACT. A sequence $S=\{s_n\}$ is said to be a periodic set of period τ ($\neq 0$) if and only if $S^*=\{s_n+\tau\}$ can be rearranged to be a sequence to coincide with S. Let F be the class of all entire functions f satisfying the growth condition:

$$\limsup_{r\to\infty} \log\log\log M(r,f)/\log r < 1.$$

In this paper it is shown that if $f\in F$ and the zero sets of f and f' both are periodic sets with the same period τ, then f can be expressed as $f(z)=e^{cz}g(z)$, where c is a constant and $g(z)$ is a periodic entire function with period τ. A counterexample is exhibited to show that the above condition is a necessary one.

1. **Introduction.** Let $S=\{s_n\}$ denote a sequence of complex numbers. S is called a periodic set with period τ ($\neq 0$) iff $S^*=\{s_n+\tau\}$ can be rearranged to be a sequence to coincide with S. Accordingly, an empty set is a periodic set. Let f denote an entire function and $Z_{f(z)}$ denote the zero set of $f(z)$. Clearly, if f is a periodic function with period τ, then all the sets $Z_{f^{(n)}(z)}$, $n=0, 1, 2, \cdots$, are periodic sets with the same period τ. However, the converse of the above result is not true. That is, if all the zero sets $Z_{f^{(n)}}$, $n=0, 1, 2, \cdots$, of an entire function f are periodic sets with the same period τ, it does not follow that f is an entire function with period τ. To verify this, choose any complex number $b\neq$rational number and $f(z)=e^{ibz}\sin z$. Then it is easy to see that all the zero sets $Z_{f^{(n)}(z)}$, $n=0, 1, 2, \cdots$, are periodic sets with the same period but that $f(z)$ fails to be a periodic function. More generally, for any constant c and periodic entire function $g(z)$, the function f of the form

$$f(z) = e^{cz}g(z) \tag{1}$$

has the property that all the zero sets $Z_{f^{(n)}(z)}$, $n=0, 1, 2, \cdots$, are periodic sets with the same period.

It becomes natural for us to ask the following question: Does form (1) include all entire functions f such that all zero sets $Z_{f^{(n)}(z)}$, $n=0, 1, 2, \cdots$, are periodic sets with the same period?

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We are unable to answer this question completely. However, if one puts a certain growth restriction on the functions to be considered, then a stronger result can be obtained as follows:

THEOREM. *Let f be an entire function with*

$$\limsup_{r\to\infty} \log\log\log M(r,f)/\log r < 1. \tag{2}$$

Then f has the form (1) *if and only if the zero sets* $Z_{f(z)}$ *and* $Z_{f'(z)}$ *are periodic sets with the same period* τ.

REMARKS. (A) The Theorem is true for any entire function of finite order.

(B) The Theorem may be false if condition (2) is violated. There is no difficulty in showing that $f(z)=\exp(e^{z/2}-e^{z/\sqrt{2}})$ is a counterexample. We omit the verification here.

PROOF. The necessity is obvious. We proceed to the sufficiency. By assumption, we have, for some complex number $\tau\neq 0$,

$$f(z)/f(z+\tau) = e^{\alpha(z)}, \tag{3}$$

and

$$f'(z)/f'(z+\tau) = e^{\beta(z)}, \tag{4}$$

where $\alpha(z)$ and $\beta(z)$ are entire functions. We will now show that α' is identically zero. Suppose that $\alpha'(z)\not\equiv 0$. Then from equation (3) we have

$$f(z) = f(z+\tau)e^{\alpha(z)}, \tag{5}$$

and hence

$$f'(z) = f'(z+\tau)e^{\alpha(z)} + \alpha'(z)f(z+\tau)e^{\alpha(z)}. \tag{6}$$

Thus combining equations (5) and (6) and using equation (4), equation (6) becomes

$$f'(z+\tau)[e^{\beta(z)} - e^{\alpha(z)}] = \alpha'(z)f(z+\tau)e^{\alpha(z)}. \tag{7}$$

Then replacing z by $z+\tau$ in equation (7),

$$f'(z+2\tau)[e^{\beta(z+\tau)} - e^{\alpha(z+\tau)}] = \alpha'(z+\tau)f(z+2\tau)e^{\alpha(z+\tau)}. \tag{8}$$

Now assume that $e^{\beta(z+\tau)}-e^{\alpha(z+\tau)}\not\equiv 0$; otherwise $\alpha'(z+\tau)\equiv 0$, giving a contradiction. Thus

$$\frac{f'(z+\tau)}{f'(z+2\tau)}\frac{e^{\beta(z)}-e^{\alpha(z)}}{e^{\beta(z+\tau)}-e^{\alpha(z+\tau)}} = \frac{\alpha'(z)}{\alpha'(z+\tau)}\frac{f(z+\tau)}{f(z+2\tau)}e^{\alpha(z)-\alpha(z+\tau)}. \tag{9}$$

By substituting equations (3) and (4) into equation (9),

$$e^{\beta(z+\tau)}\frac{e^{\beta(z)}-e^{\alpha(z)}}{e^{\beta(z+\tau)}-e^{\alpha(z+\tau)}}=\frac{\alpha'(z)}{\alpha'(z+\tau)}e^{\alpha(z)}. \tag{10}$$

We set $\alpha'(z)/\alpha'(z+\tau)=h(z)$ and so deduce

$$e^{\beta(z)}-e^{\alpha(z)}=h(z)e^{\alpha(z)-\beta(z+\tau)}(e^{\beta(z+\tau)}-e^{\alpha(z+\tau)}), \tag{11}$$

and hence

$$1+h(z)=e^{\beta(z)-\alpha(z)}+h(z)e^{\alpha(z+\tau)-\beta(z+\tau)}. \tag{12}$$

To finish the proof, we need to estimate the growth of $h(z)$. We first recall (cf. [**2**, p. 216]), that the order of the product or the quotient of two meromorphic functions f_1 and f_2 of order λ_1 and λ_2 respectively is $\leqq\max(\lambda_1, \lambda_2)$, and equality holds provided $\lambda_1\neq\lambda_2$. We assert now that the order of h must be less than 1, since otherwise either the order of $\alpha'(z)$ (hence $\alpha(z)$) or that of $\alpha'(z+\tau)$ (hence $\alpha(z+\tau)$) must be greater than or equal to 1. It then follows from equation (3) that the function $f(z)/f(z+\tau)$ grows at least as fast as $\exp e^{r^{1-\varepsilon}}$ for any given positive small number $\varepsilon>0$. This implies that either the function $f(z)$ (hence $f(z+\tau)$) or the function $f(z+\tau)$ (hence $f(z)$) grows at least as fast as $\exp e^{r^{1-\varepsilon}}$. This will contradict condition (2) by choosing ε sufficiently small. Now we consider two cases: (i) $\beta(z)-\alpha(z)$ is a nonconstant entire function, and (ii) $\beta(z)-\alpha(z)$ is a constant. In case (i) we have from equation (12) that

$$\delta(e^{\beta(z)-\alpha(z)}, 0)=\delta(e^{\alpha(z+\tau)-\beta(z+\tau)}h-(1+h(z)), 0)=1, \tag{13}$$

where $\delta(f, a)$ denotes the Nevanlinna deficiency for the function f at the value a. For a standard reference for this notion see [**1**].

By noting that $\delta(e^{\alpha(z+\tau)-\beta(z+\tau)}h, 0)=1$ and using the analog result of Nevanlinna's second fundamental theorem for deficient functions [**1**, p. 47], we have to conclude that $1+h(z)\equiv 0$; otherwise, $e^{\alpha(z+\tau)-\beta(z+\tau)}h$ would have three deficient functions: 0, $1+h$, and ∞ of slower order of deficiency 1. It follows that $\alpha'(z)\equiv-\alpha'(z+\tau)$, since $\alpha'(z)/\alpha'(z+\tau)\equiv-1$. Hence $\alpha'(z)$ has an order at least 1 which will lead to a contradiction, as we analysed before.

In case (ii) we have from equation (12) again that

$$1+h(z)=e^c+h(z)e^{-c}, \tag{14}$$

where c (=constant)$=\beta(z)-\alpha(z)$.

Clearly, if $h(z)=d$ (=constant, which cannot be zero!), then two cases may arise: (a) $\alpha'(z)$ is a constant, and (b) $\alpha'(z)=d\alpha'(z+\tau)$, $\alpha'\not\equiv$constant.

Case (a) yields $h(z)\equiv 1$ and hence it follows from this and equation (14) that $e^c=1$, (i.e., $e^\alpha=e^\beta$), then, according to equation (7), $\alpha'(z)\equiv 0$, giving a contradiction. Case (b) indicates that $\alpha'(z)$ has an order at least 1, which would lead to a contradiction, as before. Now if $h(z)\not\equiv$constant, then from equation (14) we easily deduce that $e^c=1$, which is impossible, as we argued in case (a). Thus we have to conclude that $\alpha'(z)\equiv 0$, and hence $\alpha(z)\equiv$constant.

Now going back to equation (3) we consider

$$f(z) = e^\alpha f(z+\tau), \tag{15}$$

where α is a constant.

Let b be any complex number such that $f(b+\tau)\neq 0$ and choose $a=\log[f(b)/f(b+\tau)]/-\tau$. Let us consider the expression

$$f(z) = e^{az}g(z), \tag{16}$$

where $g(z)$ is an entire function. We are going to show that $g(z)$ is a periodic function with period τ.

From equations (15) and (16) we obtain

$$g(z) \equiv e^{\alpha+a\tau}g(z+\tau), \tag{17}$$

which becomes

$$g(z) \equiv g(z+\tau)$$

using the assumed choice of a. This also completes the proof of the Theorem.

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