ON PERIODICITY OF ENTIRE FUNCTIONS

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Abstract. A sequence \( S = \{s_n\} \) is said to be a periodic set of period \( \tau \) \((\neq 0)\) if and only if \( S^* = \{s_n + \tau\} \) can be rearranged to be a sequence to coincide with \( S \). Let \( F \) be the class of all entire functions \( f \) satisfying the growth condition:

\[
\limsup_{r \to \infty} \frac{\log \log \log M(r,f)}{\log r} < 1.
\]

In this paper it is shown that if \( f \in F \) and the zero sets of \( f \) and \( f' \) both are periodic sets with the same period \( \tau \), then \( f \) can be expressed as \( f(z) = e^{cz} g(z) \), where \( c \) is a constant and \( g(z) \) is a periodic entire function with period \( \tau \). A counterexample is exhibited to show that the above condition is a necessary one.

1. Introduction. Let \( S = \{s_n\} \) denote a sequence of complex numbers. \( S \) is called a periodic set with period \( \tau \) \((\neq 0)\) if \( S^* = \{s_n + \tau\} \) can be rearranged to be a sequence to coincide with \( S \). Accordingly, an empty set is a periodic set. Let \( f \) denote an entire function and \( Z_{f(z)} \) denote the zero set of \( f(z) \). Clearly, if \( f \) is a periodic function with period \( \tau \), then all the sets \( Z_{f(n)(z)} \), \( n = 0, 1, 2, \ldots \), are periodic sets with the same period \( \tau \). However, the converse of the above result is not true. That is, if all the zero sets \( Z_{f(n)(z)} \), \( n = 0, 1, 2, \ldots \), of an entire function \( f \) are periodic sets with the same period \( \tau \), it does not follow that \( f \) is an entire function with period \( \tau \). To verify this, choose any complex number \( b \neq \text{rational number} \) and \( f(z) = e^{bz} \sin z \). Then it is easy to see that all the zero sets \( Z_{f(n)(z)} \), \( n = 0, 1, 2, \ldots \), are periodic sets with the same period but that \( f(z) \) fails to be a periodic function. More generally, for any constant \( c \) and periodic entire function \( g(z) \), the function \( f \) of the form

\[
f(z) = e^{cz} g(z)
\]

has the property that all the zero sets \( Z_{f(n)(z)} \), \( n = 0, 1, 2, \ldots \), are periodic sets with the same period.

It becomes natural for us to ask the following question: Does form (1) include all entire functions \( f \) such that all zero sets \( Z_{f(n)(z)} \), \( n = 0, 1, 2, \ldots \), are periodic sets with the same period?
We are unable to answer this question completely. However, if one puts a certain growth restriction on the functions to be considered, then a stronger result can be obtained as follows:

**Theorem.** Let $f$ be an entire function with

$$\limsup_{r \to \infty} \frac{\log \log \log M(r, f)}{\log r} < 1.$$  \hspace{1cm} (2)

Then $f$ has the form (1) if and only if the zero sets $Z(f,z)$ and $Z(r,z)$ are periodic sets with the same period $\tau$.

**Remarks.** (A) The Theorem is true for any entire function of finite order.
(B) The Theorem may be false if condition (2) is violated. There is no difficulty in showing that $f(z) = \exp(e^{z/2} - e^{z/\sqrt{2}})$ is a counterexample. We omit the verification here.

**Proof.** The necessity is obvious. We proceed to the sufficiency. By assumption, we have, for some complex number $\tau \neq 0$,

$$f(z)/f(z + \tau) = e^{\alpha(z)},$$  \hspace{1cm} (3)

and

$$f'(z)/f'(z + \tau) = e^{\beta(z)},$$  \hspace{1cm} (4)

where $\alpha(z)$ and $\beta(z)$ are entire functions. We will now show that $\alpha'$ is identically zero. Suppose that $\alpha'(z) \neq 0$. Then from equation (3) we have

$$f(z) = f(z + \tau) e^{\alpha(z)},$$  \hspace{1cm} (5)

and hence

$$f'(z) = f'(z + \tau) e^{\alpha(z)} + \alpha'(z) f(z + \tau) e^{\alpha(z)}.$$  \hspace{1cm} (6)

Thus combining equations (5) and (6) and using equation (4), equation (6) becomes

$$f'(z + \tau) [e^{\beta(z)} - e^{\alpha(z)}] = \alpha'(z) f(z + \tau) e^{\alpha(z)}.$$  \hspace{1cm} (7)

Then replacing $z$ by $z + \tau$ in equation (7),

$$f'(z + 2\tau) [e^{\beta(z+\tau)} - e^{\alpha(z+\tau)}] = \alpha'(z + \tau) f(z + 2\tau) e^{\alpha(z+\tau)}.$$  \hspace{1cm} (8)

Now assume that $e^{\beta(z+\tau)} - e^{\alpha(z+\tau)} \neq 0$; otherwise $\alpha'(z + \tau) \equiv 0$, giving a contradiction. Thus

$$\frac{f'(z + \tau)}{f'(z + 2\tau)} \frac{e^{\beta(z)} - e^{\alpha(z)}}{e^{\beta(z+\tau)} - e^{\alpha(z+\tau)}} = \frac{\alpha'(z)}{\alpha'(z + \tau)} \frac{f(z + \tau)}{f(z + 2\tau)} e^{\alpha(z) - \alpha(z+\tau)}.$$  \hspace{1cm} (9)
By substituting equations (3) and (4) into equation (9),
\[ e^{\beta(z+\tau)} - e^{\alpha(z)} = \frac{\alpha'(z)}{\alpha'(z+\tau)} e^{\alpha(z)}. \]

We set \( \alpha'(z)/\alpha'(z+\tau) = h(z) \) and so deduce
\[ e^{\beta(z)} - e^{\alpha(z)} = h(z)e^{\alpha(z)-\beta(z+\tau)}(e^{\beta(z+\tau)} - e^{\alpha(z+\tau)}), \]
and hence
\[ 1 + h(z) = e^{\beta(z)-\alpha(z)} + h(z)e^{\alpha(z+\tau)-\beta(z+\tau)}. \]

To finish the proof, we need to estimate the growth of \( h(z) \). We first recall (cf. [2, p. 216]), that the order of the product or the quotient of two meromorphic functions \( f_1 \) and \( f_2 \) of order \( \lambda_1 \) and \( \lambda_2 \) respectively is \( \leq \max(\lambda_1, \lambda_2) \), and equality holds provided \( \lambda_1 \neq \lambda_2 \). We assert now that the order of \( h \) must be less than 1, since otherwise either the order of \( \alpha'(z) \) (hence \( \alpha(z) \)) or that of \( \alpha'(z+\tau) \) (hence \( \alpha(z+\tau) \)) must be greater than or equal to 1. It then follows from equation (3) that the function \( f(z) = e^{\beta(z)} - e^{\alpha(z)} \) grows at least as fast as \( \exp \varepsilon e^{-z} \) for any given positive small number \( \varepsilon > 0 \). This implies that either the function \( f(z) \) (hence \( f(z+\tau) \)) or the function \( f(z+\tau) \) (hence \( f(z) \)) grows at least as fast as \( \exp e^{1-z} \). This will contradict condition (2) by choosing \( \varepsilon \) sufficiently small. Now we consider two cases: (i) \( \beta(z) - \alpha(z) \) is a nonconstant entire function, and (ii) \( \beta(z) - \alpha(z) \) is a constant. In case (i) we have from equation (12) that
\[ \delta(e^{\beta(z)-\alpha(z)},0) = \delta(e^{\alpha(z+\tau)-\beta(z+\tau)}h - (1 + h(z)), 0) = 1, \]
where \( \delta(f, a) \) denotes the Nevanlinna deficiency for the function \( f \) at the value \( a \). For a standard reference for this notion see [1].

By noting that \( \delta(e^{\alpha(z+\tau)-\beta(z+\tau)}h, 0) = 1 \) and using the analog result of Nevanlinna's second fundamental theorem for deficient functions [1, p. 47], we have to conclude that \( 1 + h(z) \equiv 0 \); otherwise, \( e^{\alpha(z+\tau)-\beta(z+\tau)}h \) would have three deficient functions: 0, 1 + h, and \( \infty \) of slower order of deficiency 1. It follows that \( \alpha'(z) \equiv -\alpha'(z+\tau) \), since \( \alpha'(z)/\alpha'(z+\tau) \equiv -1 \). Hence \( \alpha'(z) \) has an order at least 1 which will lead to a contradiction, as we analysed before.

In case (ii) we have from equation (12) again that
\[ 1 + h(z) = e^c + h(z)e^{-c}, \]
where \( c \) (constant) \( = \beta(z) - \alpha(z) \).

Clearly, if \( h(z) = d \) (constant, which cannot be zero!), then two cases may arise: (a) \( \alpha'(z) \) is a constant, and (b) \( \alpha'(z) = d\alpha'(z+\tau), \alpha' \neq \text{constant} \).
Case (a) yields $h(z) \equiv 1$ and hence it follows from this and equation (14) that $e^z = 1$, (i.e., $e^z = e^\beta$), then, according to equation (7), $\alpha'(z) \equiv 0$, giving a contradiction. Case (b) indicates that $\alpha'(z)$ has an order at least 1, which would lead to a contradiction, as before. Now if $h(z)$ is not constant, then from equation (14) we easily deduce that $e^z = 1$, which is impossible, as we argued in case (a). Thus we have to conclude that $\alpha'(z) \equiv 0$, and hence $\alpha(z) \equiv \text{constant}$.

Now going back to equation (3) we consider

$$f(z) = e^z f(z + \tau),$$

where $\alpha$ is a constant.

Let $b$ be any complex number such that $f(b + \tau) \neq 0$ and choose $a = \log[f(b)/f(b + \tau)] - \tau$. Let us consider the expression

$$f(z) = e^{az} g(z),$$

where $g(z)$ is an entire function. We are going to show that $g(z)$ is a periodic function with period $\tau$.

From equations (15) and (16) we obtain

$$g(z) \equiv e^{az} g(z + \tau),$$

which becomes

$$g(z) \equiv g(z + \tau)$$

using the assumed choice of $a$. This also completes the proof of the Theorem.

REFERENCES
