A PROPERTY OF TRANSFERABLE LATTICES

G. GRÄTZER

Abstract. A lattice $K$ is transferable if whenever $K$ can be embedded into the ideal lattice of a lattice $L$, then $K$ can be embedded in $L$. An element is called doubly reducible if it is both join- and meet-reducible. In this note it is proved that every lattice can be embedded into the ideal lattice of a lattice with no doubly reducible element. It follows from this result that a transferable lattice has no doubly reducible element.

1. Introduction. In [5] two concepts of transferability of a lattice were introduced, named transferability and weak transferability in [2] and named sharp transferability and transferability, respectively, in this paper and [4]. A rather satisfying theory of sharp transferability can be found in [2] and [4]; see also [3] for the case of semilattices. Recently K. Baker proved that all finite projective lattices are transferable (see [1]). Still, the only known property of transferable lattices is the one announced in [5] without proof. The purpose of this note is to supply a proof of this property (see Theorem below).

First, two definitions. A lattice $K$ is called weakly transferable iff whenever $K$ can be embedded into the lattice of all ideals of a lattice $L$, then $K$ can also be embedded into $L$. Observe, that in the papers referred to above the finiteness of $K$ is also assumed; for the purposes of this note, however, it is not necessary to assume that $K$ is finite. An element $a$ of the lattice $K$ is doubly reducible iff there exist elements $x, y, z, u$ of $K$, all distinct from $a$, such that $a = x \vee y = z \wedge u$.

Theorem. A transferable lattice contains no doubly reducible element.

2. Proof of the Theorem. Let $A$ and $B$ be posets. The lexicographic product of $A$ and $B$, denoted by $A \otimes B$, is a poset defined on $A \times B$ with the ordering $(a, a' \in A, b, b' \in B)$:

$$(1) \quad (a, b) \preceq (a', b') \iff a < a' \quad \text{or} \quad a = a' \quad \text{and} \quad b \leq b'.$$

In this note, let $A$ and $B$ be lattices and let $B$ have a least element 0 and largest element 1.

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Lemma 1. \( A \times B \) is a lattice and, for \( a, a' \in A, b, b' \in B \), we have that

\[
(2) \quad \langle a, b \rangle \lor \langle a', b' \rangle = \langle a \lor a', b'' \rangle
\]

with suitable \( b'' \in B \); in fact, if \( a \) and \( a' \) are incomparable, then \( b'' = 0 \).

Proof. Trivial; observe, that if \( a < a' \), then \( b'' = b' \); if \( a' < a \), then \( b'' = b \); if \( a = a' \), then \( b'' = b \lor b' \).

Call an element \( p \) join-reducible if \( p = x \lor y \) with \( p \neq x \) and \( p \neq y \); otherwise \( p \) is join-irreducible. The dual concepts are meet-reducible and meet-irreducible. From Lemma 1 we conclude immediately:

Corollary 2. All join-reducible elements of \( A \times B \) are of the form \( \langle a, b \rangle \) where \( b = 0 \) or \( b \) is join-reducible in \( B \).

Corollary 2 and its dual yield:

Corollary 3. Let us assume that \( B \) has more than one element, and \( 0 \) is meet-irreducible, and \( 1 \) is join-irreducible in \( B \). Then all doubly reducible elements of \( A \times B \) have the form \( \langle a, b \rangle \), where \( b \) is doubly reducible in \( B \). In particular, if \( B \) has no doubly reducible element, then neither does \( A \times B \).

Now we map ideals of \( A \) into ideals of \( A \times B \). Let \( I \) be an ideal of \( A \). We set

\[
(3) \quad I = \{ \langle a, b \rangle \mid a \in I, b \in B \}.
\]

Lemma 4. For any ideal \( I \) of \( A \), the set \( I \) is an ideal of \( A \times B \). The map \( I \rightarrow I \) is one-to-one, and for ideals \( I \) and \( J \) of \( A \) it satisfies

\[
(4) \quad I \land J = (I \land J)^-;
\]

it also satisfies

\[
(5) \quad I \lor J = (I \lor J)^-,
\]

provided that \( I \lor J \) is not a principal ideal.

Proof. It follows immediately from (1) and (2) that (3) defines an ideal. Now, \( \langle a, b \rangle \in I \land J \) iff \( \langle a, b \rangle \in I \) and \( \langle a, b \rangle \in J \), which is, by (3), equivalent to \( a \in I \) and \( a \in J \), that is, to \( a \in I \land J \), which means that

\[
\langle a, b \rangle \in (I \land J)^-,
\]

proving (4).

By (4), the inclusion "\( \subseteq \)" is obvious in (5). To prove the reverse inclusion, let

\[
\langle a, b \rangle \in (I \lor J)^-.
\]
Then \( a \in I \lor J \) by (3); since, by hypothesis, \( I \lor J \) is not principal, there exists an \( a' \in I \lor J \) satisfying \( a < a' \). Also, since \( a' \in I \lor J \), we get elements \( i, j \) of \( A \) with \( a' \leq i \lor j \), \( i \in I \), \( j \in J \). By (3), \( \langle i, 0 \rangle \in I \), \( \langle j, 0 \rangle \in J \), and so, using (1) and (2),
\[
\langle a, b \rangle < \langle i \lor j, 0 \rangle = \langle i, 0 \rangle \lor \langle j, 0 \rangle \in I \lor J,
\]
proving the reverse inclusion, and thus Lemma 4.

Next, we need a trivial construction.

**Lemma 5.** Let \( K \) be an arbitrary lattice. \( K \) has an embedding \( \varphi \) into the lattice of all ideals of a suitable lattice \( L \) such that, for all \( a \in K \), \( a \varphi \) is a nonprincipal ideal of \( L \).

**Proof.** For instance, let \( N \) be the chain of natural numbers, \( L = K \times N \), and, for \( a \in K \), set \( a \varphi = \{ \langle x, n \rangle \mid x \leq a \} \).

Combining the embeddings of Lemma 4 (with, say, \( B \) the two-element chain) and Lemma 5 we obtain the main result of this note:

**Theorem 6.** Every lattice \( K \) can be embedded into the lattice of all ideals of some lattice \( L \) with no doubly reducible element.

The Theorem of the Introduction follows immediately from Theorem 6. Indeed, if \( K \) is a transferable lattice, then we embed \( K \) into \( I(L) \) by Theorem 6, where \( L \) is a lattice with no doubly reducible element. By transferability, \( K \) can be embedded into \( L \); hence \( K \) has no doubly reducible element.

**References**


Department of Mathematics, University of Manitoba, Winnipeg R3T 2N2, Manitoba, Canada