ABSTRACT. In this paper we give an alternative proof, without reference to Urysohn's lemma, of the metrization theorem of Bing [2], Nagata [6], and Smirnov [8] via the theory of symmetric spaces as developed by H. Martin in [5].

A symmetric $d$ on a point set $X$ is a function $X \times X \to [0, \infty)$ satisfying
1. $d(x, y) = 0$ if and only if $x = y$, and
2. $d(x, y) = d(y, x)$. A topology $T$ on $X$ is said to be determined by $d$ provided that for every subset $U$ of $X$, $U$ belongs to $T$ if and only if it contains an $\varepsilon$-sphere $N(p; \varepsilon) = \{x : d(p, x) < \varepsilon\}$ about each of its points $p$. The data $X$, $d$, and $T$ is called a symmetric space. Such a space need not be Hausdorff or first countable, but H. W. Martin [5] has proved the theorem below.

**Theorem 1.** Let $X$ be a topological space symmetrizable via a symmetric $d$. If $d(K, F) > 0$ whenever $K \cap F = \emptyset$, $K$ is compact, and $F$ closed, then $X$ is metrizable.

This theorem strengthened an earlier theorem of A. V. Arhangel'skiï [1], who introduced the notion of symmetric spaces. Martin achieves a proof of Theorem 1 by showing that $X$ must satisfy the hypotheses of Mrs. Frink's theorem [3], a classical result in metrization theory. As a corollary of Theorem 1, Martin (and Arhangel'skiï) obtains the theorem of S. Hanai and K. Morita [4], and A. H. Stone [9] on the metrizability of perfect images of metric spaces.

The purpose of this paper is to obtain the metrization theorem of Bing [2], Nagata [6], and Smirnov [8] as a consequence of Theorem 1. It is interesting to note that Urysohn's lemma is never used in this approach, as was the case in the approach used by D. Rolfsen in [7]. More specifically, let us assume that $X$ is a regular, $T_1$ space with a $\sigma$-locally finite base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n$ is locally finite and $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$, $n \geq 1$. 

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For \( x, y \in X \), \( x \neq y \), put \( m(x, y) = \min \{ n : \exists B \in \mathcal{B}_n \text{ with } x \in B, y \notin B \} \), \( t(x, y) = 1/m(x, y) \), and \( d(x, y) = \max \{ t(x, y), t(y, x) \} \). Also, put \( d(x, x) = 0 \). Then we shall prove the following theorem.

**Theorem 2.** \( X \) is symmetrizable via \( d \). Furthermore, \( d(K, F) > 0 \) whenever \( K \cap F = \emptyset \), \( K \) is compact, and \( F \) closed. Therefore, \( X \) is metrizable.

**Proof.** Denote by \( T \) and \( T_d \) the given and \( d \)-induced topologies on \( X \), respectively. We must show that (1) \( T \subset T_d \), (2) \( T_d \subset T \), and (3) \( d(K, F) > 0 \) whenever \( K \cap F = \emptyset \), \( K \) is compact, and \( F \) closed.

To establish (1), assume that \( B \in \mathcal{B} \), \( x \in B \). Choose \( B_x \in \mathcal{B} \) such that \( x \in B_x \subset B \). If \( B_x \in \mathcal{B}_n \), we have \( N(x; 1/n) \subset B_x \subset B \), so that \( B \) is open in \( T_d \).

To establish (2), let \( F \) be a \( T_d \)-closed set. If \( F \) is not \( T \)-closed (\( X \) is first countable because of \( \sigma \)-locally finite \( \mathcal{B} \)), there is a point \( x \notin F \) and a sequence \( x_1, x_2, \cdots \) of points in \( F \) converging to \( x \). We shall show that

(i) \( \lim_{i \to \infty} t(x, x_i) = 0 \),
(ii) \( \inf \{ t(x, x_i) : i \geq 1 \} = 0 \), so that
(iii) \( \inf \{ d(x, x_i) : i \geq 1 \} = 0 \) holds, which contradicts \( d(x, F) > 0 \).

To this end, let \( x \in B \in \mathcal{B}_n \). Denote by \( U \) the intersection of all members of \( \mathcal{B}_n \) containing \( x \). There exists a positive integer \( N \) satisfying \( x_i \in U \) for \( i \geq N \), whence \( t(x, x_i) < 1/n \). Since \( n \) can be chosen as large as we please, (i) follows.

As for (ii), let \( x \in B \in \mathcal{B}_n \). Denote by \( V \) an open neighborhood of \( x \) that intersects only finitely many members of \( \mathcal{B}_n \) and satisfies \( V \subset B \). Choose \( N \) so that \( x_i \in V \) for \( i \geq N \). Whenever \( i \geq N \), let \( U_i \) represent the intersection of all members of \( \mathcal{B}_n \) containing \( x_i \). It follows that for infinitely many such values of \( i \), the sets \( U_i \) are identical, there being only finitely many such intersections. Denoting such a common value by \( U \), it is clear that \( x \in U \), and therefore that \( t(x, x_i) < 1/n \). This establishes (ii), (iii), and (2).

To establish (3), let \( K \) be compact, \( F \) closed, and \( K \cap F = \emptyset \). Let \( B_1, B_2, \cdots, B_k \) be a finite cover of \( K \) by members of \( \mathcal{B} \) with \( B_i \cap F = \emptyset \), \( i = 1, \cdots, k \). Choose \( n \) such that \( B_i \in \mathcal{B}_n \), \( i = 1, \cdots, k \). Then we have \( 0 < 1/n \leq t(K, F) \leq d(K, F) \).

**References**


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