# A PROBLEM OF MARTIN CONCERNING STRONGLY CONVEX METRICS ON $E^3$

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ABSTRACT. If $d$ is a strongly convex metric on $E^3$ and $C$ is a simple closed curve in $E^3$ such that $C$ is the union of three line segments then $C$ is unknotted.

In a talk presented to a Topology Conference at Arizona State University in 1967, Joseph Martin asked the following question: If $d$ is a strongly convex metric on Euclidean 3-space $E^3$ and $C$ is a simple closed curve in $E^3$ that is the union of three line segments (with respect to the metric $d$) is $C$ unknotted? The purpose of this note is to answer his question in the affirmative.

A *partially ordered space* is a space $X$ together with a partial order $\leq$ on $X$ such that $\leq$ has a closed graph.

For $x \in X$ we let

$$L(x) = \{y \in X \,|\, y \leq x\} \quad \text{and} \quad M(x) = \{y \in x \,|\, x \leq y\}.$$

A *chain* is a totally ordered set. An *order arc* is a compact and connected chain. An *antichain* is a set which contains no nondegenerate chain. A set $A$ is a maximal antichain of $X$ if $A$ is an antichain and for each $x \in X$, there exists $y \in A$ such that either $x \leq y$ or $y \leq x$. An element $\theta \in X$ is called the *zero* (resp. *identity*) of $X$ if $\theta \leq x$ (resp. $x \leq \theta$) for each $x \in X$.

We shall need the following result which is a slight generalization of Theorem 2.6 in [5].

THEOREM 1. *Let $X$ be a compact metric partially ordered space such that $X$ has a zero and an identity and, for each $x \in X$, $L(x) \cup M(x)$ is a connected set. If $A$ is a compact antichain in $X$ then $A$ is contained in a compact maximal antichain of $X$.*

PROOF. Let $A$ be a compact antichain in $X$ which is neither the zero nor the identity of $X$. Let $Y$ be the quotient space obtained from $X$ by

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identifying the set $A$ to a point and let $\pi$ be the natural projection of $X$ onto $Y$.

Let $\leqq'$ be the smallest partial order on $Y$ that makes $\pi$ an order preserving function, i.e. for $x \in X$ let

$$L(\pi(x)) = \pi(L(x)) \quad \text{if } A \cap L(x) = \varnothing$$
$$= \bigcup \{\pi(L(y)) \,|\, y \in A \cup \{x\}\} \quad \text{otherwise.}$$

Then $\leqq'$ has a closed graph, $Y$ has a zero and an identity, and, for each $y \in Y$, $L(y) \cup M(y)$ is connected. By [5, Theorem 2.6] there exists a compact maximal antichain $C$ of $Y$ such that $\pi(A) \subset C$. Hence, $\pi^{-1}(C)$ is a compact maximal antichain of $X$ which contains $A$.

LEMMA 2. *Let $B$ denote the closed unit ball with center at the origin $\theta$ in $E^3$. Let $A$ be an arc in $B$ such that, for each $\varepsilon$ with $0 < \varepsilon \leqq 1$, $A$ meets the 2-sphere $S_\varepsilon$ with center at the origin and radius $\varepsilon$ in exactly two points. Then there is a homeomorphism $h$ of $B$ onto $B$ such that $h$ carries each $S_\varepsilon$ onto itself and $h$ carries $A$ onto a diameter of $B$.*

PROOF. Let $A_+$ and $A_-$ be the two half arcs determined by $A$, i.e. $A = A_+ \cup A_-$, $A_+ \cap A_- = \theta$. Let $\phi: (0, 1] \to S_1$ be the map determined by $A_+$, i.e., $\phi(\varepsilon) = \pi_\varepsilon(A_+ \cap S_\varepsilon)$, where $\pi_\varepsilon: S_\varepsilon \to S_1$ is the natural radial projection. Since the orthogonal group $O(3)$ is a bundle over the 2-sphere $S_1$ (see [4, p. 33]), $\phi$ lifts to a map $\Phi: (0, 1] \to O(3)$. Setting

$$h_1\big|_{S_\varepsilon} = \pi_\varepsilon^{-1} \circ (\Phi(\varepsilon)\big|_{S_1})^{-1} \circ \pi_\varepsilon, \quad h_1(\theta) = \theta,$$

defines a homeomorphism $h_1$ of $B$ onto itself which takes $A_+$ onto a radius. Let $R^2$ be the Euclidean plane. The complement of $h_1(A_+)$ in $B$ is homeomorphic to $(0, 1] \times R^2$ by a homeomorphism which carries $S_\varepsilon$ onto $\{\varepsilon\} \times R^2$ for each $\varepsilon$ in $(0, 1]$. Let $h_2$ be the homeomorphism of $B$ onto itself (fixed on $h_1(A_-)$) which, by translating each $\{\varepsilon\} \times R^2$ onto itself according to $h_1(A_-)$, takes $h_1(A_-)$ onto the opposite radius. Now $h = h_2 \circ h_1$ satisfies the conclusion of the lemma.

THEOREM 3. *Let $\leqq$ be a partial order with closed graph on $E^3$ such that $\theta$ is the zero of $E^3$ and, for each $x \in E^3$, $L(x)$ is an order arc. If $A$ is a simple closed curve in $E^3$ such that $A = A_1 \cup A_2 \cup A_3$ where $A_1$ and $A_2$ are order arcs, $A_1 \cap A_2 = \{\theta\}$ and $A_3$ is an antichain, then $A$ is tame and unknotted.*

PROOF. Let $p \notin E^3$ and let $S^3 = E^3 \cup \{p\}$ be the one-point compactification of $E^3$. Extend the partial order $\leqq$ on $E^3$ to a partial order (which we again denote by $\leqq$) with closed graph on $S^3$ by making $p$ the largest element of $S^3$. Then $S^3$ with this partial order satisfies the hypotheses of Theorem 1.

By Theorem 1 there is a compact maximal antichain $C$ of $S^3$ such that $A_3 \subset C$. By the proof of Theorem 3 in [6] every compact maximal antichain of $E^3 - \{\theta\}$ is a tame 2-sphere and there is a homeomorphism $h$ of $L(C) = \bigcup \{L(x) | x \in C\}$ onto the unit ball $B$ with center at the origin in $E^3$ such that, for each $\varepsilon$ with $0 < \varepsilon \leq 1$, $h^{-1}(S_\varepsilon)$ is a compact maximal antichain of $E^3$ where $S_\varepsilon$ is the sphere with center $\theta$ and radius $\varepsilon$ in $B$. Thus, for each $i = 1, 2$ and for each $\varepsilon$ with $0 < \varepsilon \leq 1$, $h^{-1}(S_\varepsilon) \cap A_i$ consists of exactly one point.

By Lemma 2 there is a homeomorphism $j$ of $B$ onto itself such that, for each $\varepsilon$ with $0 < \varepsilon \leq 1$, $j(S_\varepsilon) = S_\varepsilon$ and $j(h(A_1) \cup h(A_2))$ is a diameter of $B$. Now, $j(h(A))$ is the union of a diameter $j(h(A_1 \cup A_2))$ of $B$ together with an arc $j(h(A_3))$ on the boundary of $B$. Thus, $j(h(A))$ is tame and unknotted in $B$ and hence $A$ is tame and unknotted in $E^3$.

A metric $d$ for a topological space $X$ is said to be *strongly convex* if for each $x, y \in X$ there exists a unique $z \in X$ such that $d(x, y)/2 = d(x, z) = d(y, z)$.

A *line segment* in a metric space $(X, d)$ is a set isometric to a segment of the real line with its usual metric.

A metric $d$ for a compact space $X$ is strongly convex if and only if each pair of points $x$ and $y$ of $X$ is contained in a unique minimal line segment $xy$ (see [2]).

THEOREM 4. *Let $d$ be a strongly convex metric for $E^3$. If $C$ is a simple closed curve in $E^3$ such that $C$ is the union of three line segments then $C$ is tame and unknotted.*

PROOF. Suppose $C$ is composed of the three line segments $ab$, $ac$ and $bc$. Let $\leq$ be the partial order on $E^3$ defined by setting $x \leq y$ if and only if $x \in ay$. Then (see [6]) $\leq$ has a closed graph, zero $a$, and, for each $x \in E^3$, $L(x)$ is an order arc.

By the proof of Theorem 3 it follows that $ab \cup ac$ is a tame arc. Since $a$ was arbitrary, $C$ is locally tame. By [1] $C$ is tame.

Let $S = \{x \in E^3 | d(x, a) = 1\}$. By the proof of Theorem 3, $S$ is a 2-sphere and $B = L(S)$ is a 3-cell. Rolfsen proved in [3] that, for each $x \in B - \{a\}$, $M(x) \cap S$ is a proper subcontinuum of $S$ which does not separate $S$. We may suppose without loss of generality that $C \subset B - S$.

If $x, y \in bc$ such that $x < y$ then $xy \subset bc$. Let $X$ be the quotient space obtained from $B$ by identifying each line segment $xy \subset bc$ such that $x < y$ to a point. Let $\pi$ be the natural projection of $B$ onto $X$. Then $X$ is a compact metric space. The partial order on $B$ induces in a natural way a partial order $\leq'$ on $X$ such that $\leq'$ has closed graph, zero $a$, set of maximal elements $S$; for each $x \in X - \{a\}$, $L(x)$ is a nondegenerate order arc, and, for each $x \in X - \{a\}$, $\{y \in S | x \leq' y\}$ is a proper subcontinuum of $S$

which does not separate $S$. By Theorem 5 in [6], $X$ is a 3-cell. Now, $\pi(C)$ is a simple closed curve in $X$ which satisfies the hypotheses of Theorem 3. Hence, $\pi(C)$ is tame and unknotted in $X$. Thus, the fundamental group of $X-\pi(C)$ is infinite cyclic. Since $\pi$ is a homeomorphism off of $C$, $B-C$ is homeomorphic to $X-\pi(C)$. In particular, the fundamental group of $B-C$ is infinite cyclic. Thus $C$ is unknotted in $B$ and hence in $E^3$.

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