A COMPARISON OF METRICS ON TEICHMÜLLER SPACE

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Abstract. The length in the Weil-Petersson metric of the Teichmüller geodesic between two points is computed, yielding the result that the Weil-Petersson metric is dominated by a constant multiple of the Teichmüller metric.

Let $T(G)$ denote a Teichmüller space of Riemann surfaces which arise as the quotients of the unit disc $U$ by finitely generated Fuchsian groups of the first kind $G'$, isomorphic to a fixed group $G$. We shall show that the Weil-Petersson metric on $T(G)$ is dominated by a constant multiple of the Teichmüller metric, the constant depending on the given Teichmüller space.

1. Beltrami differentials; the Teichmüller metric. Denote by $B(G)$ the space of bounded measurable complex-valued functions on $U$ with the $L_\infty$ norm $||\mu||_\infty=\sup_{z\in U}|\mu(z)|$ satisfying

$$\mu(y(z))\gamma'(z)/\gamma'(z) = \mu(z),$$

and denote by $B_1(G)$ its open unit ball. If $\mu \in B_1(G)$, let $w^\mu$ denote the unique quasiconformal mapping of the unit disc onto itself which satisfies the Beltrami equation $w_z = \mu w^\mu z$ and keeps the points 1, $i$, $-1$ fixed. If $\mu$ depends analytically on a real parameter $t$, then for all $z$ in $U$, $w^\mu(z)$ is an analytic function of $t$.

There is a natural surjection of $B_1(G)$ onto $T(G)$ defined by $\mu \mapsto P_\mu^\mu = \text{the equivalence class of } U/G^\mu$, where $G^\mu = w^\mu \circ G \circ (w^\mu)^{-1}$. Note that $w^0 = \text{identity}$ so that $P^0$ is the equivalence class of $U/G$, which we refer to as the origin of $T(G)$ and write $P = P^0$.

It is a consequence of Teichmüller's theorem that given any point $P' \in T(G)$ there exists a unique $v \in B_1(G)$ such that $P_\nu^\nu = P'$ and $v$ is of the form $v = k_\phi/|\phi|$, where $0 \leq k < 1$ and $\phi$ is a quadratic differential on $U/G$.

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\( \phi \neq 0 \). Note that \( ||v||_\infty = k \). The Teichmüller distance between \( P \) and \( P' \) is given by
\[
T(P, P') = \frac{1}{2} \log \frac{1 + k}{1 - k}.
\]
This defines a complete metric on \( T(G) \) (see [4]).

2. The complex analytic structure; the Weil-Petersson metric. Consider the composition of quasiconformal mappings, \( w^\mu \circ w^\nu = w^\rho \). One computes the following relation among the Beltrami differentials [1], [2]:
\[
\rho = \frac{\mu - \lambda}{1 - \frac{\lambda}{\mu}} w_2^\lambda : (w_\lambda)^{-1}.
\]
For fixed \( \lambda \in B_1(G) \), this equation defines \( \rho \in B_1(G^\lambda) \) as an analytic function \( \rho(\mu) \). Its Fréchet derivative at \( \lambda \) in the direction \( v \in B(G) \) is given by
\[
\lim_{t \to 0} \frac{\rho(\lambda + tv) - \rho(\lambda)}{t} = \left( \frac{v}{1 - |\lambda|^2} \right) \circ (w_\lambda)^{-1} = L^\lambda_v.
\]
Then \( L^\lambda_v \in B_1(G) \).

Let \( N(G) \) denote the subspace of \( B(G) \) consisting of Beltrami differentials satisfying
\[
\int_{\mathcal{U}/G} v(z) \phi(z) \, dx \, dy = 0
\]
for all quadratic differentials \( \phi \) on \( \mathcal{U}/G \). The space \( B(G)/N(G) \) has finite complex dimension and is used to define a complex analytic structure on \( T(G) \) as follows (see [1], [2]). Let \( v_1, \ldots, v_s \) be elements of \( B(G) \) whose equivalence classes form a complex basis of \( B(G)/N(G) \). Let \( \zeta = (\zeta_1, \ldots, \zeta_s) \in \mathbb{C}^s \), and set \( m(\zeta) = \zeta_1 v_1 + \cdots + \zeta_s v_s \). Define a mapping of the open set in \( \mathbb{C}^s \) consisting of all \( \zeta \) such that \( ||m(\zeta)||_\infty < 1 \) into \( T(G) \) by \( \zeta \mapsto P^m \), \( m = m(\zeta) \). This mapping has a nonvanishing Jacobian at the origin and so maps an open neighborhood of \( 0 \in \mathbb{C}^s \) homeomorphically onto an open neighborhood \( \mathcal{N} \) of the origin \( P \in T(G) \). Hence, if \( P^\mu \) is any point in \( \mathcal{N} \), then there exist unique complex numbers \( \zeta(\mu) = (\zeta_1(\mu), \ldots, \zeta_s(\mu)) \) such that \( \zeta(\mu) \mapsto P^\mu \). The \( \zeta(\mu) \) are complex analytic coordinates in \( \mathcal{N} \) (with respect to the basis \( v_1, \ldots, v_s \); again see [1], [2]). Note that \( P^\mu = P^m \), \( m = m(\zeta(\mu)) \), but that \( \mu \) and \( m(\zeta(\mu)) \) are not necessarily congruent modulo \( N(G) \).

Given \( \mu, \nu \in B(G) \), the Petersson inner product is defined by
\[
\langle \mu, \nu \rangle = \int_{\mathcal{U}/G} \mu(z) \overline{\nu(z)} (1 - |z|^2)^{-2} \, dx \, dy.
\]
The space $B(G)/N(G)$ may be identified with the tangent space to $T(G)$ at $P$ [2]. The corresponding Riemannian metric, called the Weil-Petersson metric, has fundamental form

$$\sum g_{ij} d\zeta_i d\overline{\zeta_j}$$

with $g_{ij}(0) = \langle v_i, v_j \rangle$ on $B(G)/N(G)$, and for $\zeta \neq 0$, $g_{ij}(\zeta) = \langle L_{v_i}^m, L_{v_j}^m \rangle$ on $B(G^m)/N(G^m)$, $m = m(\zeta)$.

Denote by $W(P, P')$ the distance between two points $P$ and $P'$ in $T(G)$ in the Weil-Petersson metric.

3. Comparison of metrics. We are now in a position to prove the following theorem.

**Theorem.** Given two points $P$ and $P'$ of a Teichmüller space $T(G)$, then

$$W(P, P') \leq AT(P, P'),$$

where $A$ is the square root of the Poincaré area of $U/G$.

**Proof.** We may assume that $P$ is the origin of $T(G)$. Let $v = k\phi |\phi|$ be the unique Teichmüller differential such that $P' = P^v$. Let $v_1 = \frac{\phi}{|\phi|}$. Since $\phi(z) \neq 0$, we have that $v_1 \notin N(G)$. Let $v_2, \ldots, v_s$ be elements of $B(G)$ such that the equivalence classes of $v_1, \ldots, v_s$ form a basis of $B(G)/N(G)$, and let $\xi(\mu)$ be the coordinate functions in a neighborhood $N$ of $P$ defined by this basis. Finally, let $C$ denote the line $C: t \rightarrow P^{tv}$, $t \in [0, 1]$, and $L(C)$ its length in the Weil-Petersson metric. ($C$ is the Teichmüller geodesic from $P$ to $P'$.) We shall determine $L(C)$.

Assume first that $C \subset N$, so that

$$\xi(tv) = (tk, 0, \ldots, 0).$$

Then it follows from (3) and (4) that

$$L(C) = \int_0^1 k(L_{v_1}^{tv}, L_{v_1}^{tv})^{1/2} dt.$$  

It is easily seen from (1) that $|L_{v_1}^{tv}| = (1 - t^2k^2)^{-1}$ so that from (2),

$$L(C) = \left( \int_{U/G^{tv}} (1 - |z|^2)^{-2} \, dx \, dy \right)^{1/2} \int_0^1 \frac{k}{1 - t^2k^2} \, dt.$$  

The left integral is the Poincaré area of the Riemann surface $U/G^{tv}$ which depends only on the isomorphism class of $G^{tv}$ and is therefore a constant for the Teichmüller space $T(G)$. Hence

$$W(P, P') \leq L(C) = AT(P, P')$$

as was to be shown.
If the line $C$ is not contained in $N$, we cover $C$ by coordinate neighborhoods as follows. At each point $P^a\nu$ of $C$, $0 \leq a \leq 1$, consider the Beltrami differential $\rho \in B_1(G^a\nu)$ defined by $w^\rho = w^\nu \circ w^a\nu$. It is easily shown that $\rho$ is the unique Teichmüller differential whose image under the surjection $B_1(G^a\nu) \to T(G^a\nu)$ is equal to $P^\nu$. $(T(G^a\nu) = T(G)$ with origin $P^a\nu.)$ Set $\mu = \rho/\|\rho\|_\infty$ and extend to a basis of $B(G^a\nu)/N(G^a\nu)$ as before. Let $N^a$ denote the coordinate neighborhood of $P^a\nu$ with respect to this basis. Since $C$ is compact, a finite number of the $N^a$ cover $C$. Let these be denoted by $N_1, \cdots, N_n$, ordered by their origins, with $N_1 = N$. Let $P^{a_0\nu}, P^{a_1\nu}, \cdots, P^{a_n\nu}$, $0 = a_0 < a_1 < \cdots < a_n = 1$, be points of $C$ such that $P^{a_j\nu} \in N_j \cap N_{j+1}$, $j = 1, \cdots, n-1$. If $C_j$ is the segment with endpoints $P^{a_{j-1}\nu}$ and $P^{a_j\nu}$, $j = 1, \cdots, n$, then $C_j \subseteq N_j$. We repeat the previous argument to evaluate $L(C_j)$. (Slightly more work is involved due to the new coordinate system.) We obtain

$$L(C_j) = A \cdot \frac{1}{2} \left( \log \frac{1 + a_{j+1}k}{1 - a_{j+1}k} - \log \frac{1 + a_jk}{1 - a_jk} \right).$$

Summing these to obtain $L(C)$, we again have the desired result.

**Bibliography**