PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, April 1974

A COMPARISON OF METRICS ON TEICHMÜLLER SPACE

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ABSTRACT. The length in the Weil-Petersson metric of the Teichmüller geodesic between two points is computed, yielding the result that the Weil-Petersson metric is dominated by a constant multiple of the Teichmüller metric.

Let $T(G)$ denote a Teichmüller space of Riemann surfaces which arise as the quotients of the unit disc U by finitely generated Fuchsian groups of the first kind G', isomorphic to a fixed group G. We shall show that the Weil-Petersson metric on $T(G)$ is dominated by a constant multiple of the Teichmüller metric, the constant depending on the given Teichmüller space.

1. **Beltrami differentials; the Teichmüller metric.** Denote by $B(G)$ the space of bounded measurable complex-valued functions on U with the L_∞ norm $\|\mu\|_\infty=\sup_{z\in U}|\mu(z)|$ satisfying

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z),$$

and denote by $B_1(G)$ its open unit ball. If $\mu\in B_1(G)$, let w^μ denote the unique quasiconformal mapping of the unit disc onto itself which satisfies the Beltrami equation $w_{\bar z}=\mu w_z$ and keeps the points $1, i, -1$ fixed. If μ depends analytically on a real parameter t, then for all z in U, $w^\mu(z)$ is an analytic function of t.

There is a natural surjection of $B_1(G)$ onto $T(G)$ defined by $\mu\mapsto P^\mu=$ the equivalence class of U/G^μ, where $G^\mu=w^\mu\circ G\circ(w^\mu)^{-1}$. Note that $w^0=$ identity so that P^0 is the equivalence class of U/G, which we refer to as the origin of $T(G)$ and write $P=P^0$.

It is a consequence of Teichmüller's theorem that given any point $P'\in T(G)$ there exists a unique $\nu\in B_1(G)$ such that $P^\nu=P'$ and ν is of the form $\nu=k\bar\phi/|\phi|$, where $0\leq k<1$ and ϕ is a quadratic differential on U/G,

Received by the editors October 18, 1972.

AMS (MOS) subject classifications (1970). Primary 32G15.

Key words and phrases. Teichmüller space, Riemann surface, Beltrami differential, quasiconformal mapping, Teichmüller metric, Weil-Petersson metric.

[1] This work is a part of the author's doctoral dissertation at Columbia University under the direction of Professor Lipman Bers.

$\phi \not\equiv 0$. Note that $\|\nu\|_\infty = k$. The Teichmüller distance between P and P' is given by

$$T(P, P') = \frac{1}{2} \log \frac{1+k}{1-k}.$$

This defines a complete metric on $T(G)$ (see [4]).

2. **The complex analytic structure; the Weil-Petersson metric.** Consider the composition of quasiconformal mappings, $w^\rho \circ w^\lambda = w^\mu$. One computes the following relation among the Beltrami differentials [1], [2]:

$$\rho = \left\{ \frac{\mu - \lambda}{1 - \bar{\lambda}\mu} \frac{w_z^\lambda}{\overline{w_z^\lambda}} \right\} \cdot (w^\lambda)^{-1}.$$

For fixed $\lambda \in B_1(G)$, this equation defines $\rho \in B_1(G^\lambda)$ as an analytic function $\rho(\mu)$. Its Fréchet derivative at λ in the direction $\nu \in B(G)$ is given by

$$\lim_{t \to 0} \frac{\rho(\lambda + t\nu) - \rho(\lambda)}{t} = \left\{ \frac{\nu}{1 - |\lambda|^2} \frac{w_z^\lambda}{\overline{w_z^\lambda}} \right\} \circ (w^\lambda)^{-1} = L_\nu^\lambda. \tag{1}$$

Then $L_\nu^\lambda \in B_1(G)$.

Let $N(G)$ denote the subspace of $B(G)$ consisting of Beltrami differentials satisfying

$$\iint_{U/G} \nu(z)\phi(z)\, dx\, dy = 0$$

for all quadratic differentials ϕ on U/G. The space $B(G)/N(G)$ has finite complex dimension and is used to define a complex analytic structure on $T(G)$ as follows (see [1], [2]). Let $\nu_1, \cdots, \nu_s$ be elements of $B(G)$ whose equivalence classes form a complex basis of $B(G)/N(G)$. Let $\zeta = (\zeta_1, \cdots, \zeta_s) \in C^s$, and set $m(\zeta) = \zeta_1\nu_1 + \cdots + \zeta_s\nu_s$. Define a mapping of the open set in C^s consisting of all ζ such that $\|m(\zeta)\|_\infty < 1$ into $T(G)$ by $\zeta \mapsto P^m$, $m = m(\zeta)$. This mapping has a nonvanishing Jacobian at the origin and so maps an open neighborhood of $0 \in C^s$ homeomorphically onto an open neighborhood N of the origin $P \in T(G)$. Hence, if P^μ is any point in N, then there exist unique complex numbers $\zeta(\mu) = (\zeta_1(\mu), \cdots, \zeta_s(\mu))$ such that $\zeta(\mu) \mapsto P^\mu$. The $\zeta(\mu)$ are complex analytic coordinates in N (with respect to the basis $\nu_1, \cdots, \nu_s$; again see [1], [2]). Note that $P^\mu = P^m$, $m = m(\zeta(\mu))$, but that μ and $m(\zeta(\mu))$ are not necessarily congruent modulo $N(G)$.

Given $\mu, \nu \in B(G)$, the Petersson inner product is defined by

$$\langle \mu, \nu \rangle = \iint_{U/G} \mu(z)\overline{\nu(z)}(1 - |z|^2)^{-2}\, dx\, dy. \tag{2}$$

The space $B(G)/N(G)$ may be identified with the tangent space to $T(G)$ at P [**2**]. The corresponding Riemannian metric, called the Weil-Petersson metric, has fundamental form

$$ds^2 = \sum g_{ij}\, d\zeta_i\, \overline{d\zeta_j} \tag{3}$$

with $g_{ij}(0)=\langle \nu_i, \nu_j\rangle$ on $B(G)/N(G)$, and for $\zeta\neq 0$, $g_{ij}(\zeta)=\langle L^m_{\nu_i}, L^m_{\nu_j}\rangle$ on $B(G^m)/N(G^m)$, $m=m(\zeta)$.

Denote by $W(P, P')$ the distance between two points P and P' in $T(G)$ in the Weil-Petersson metric.

3. **Comparison of metrics.** We are now in a position to prove the following theorem.

THEOREM. *Given two points P and P' of a Teichmüller space $T(G)$, then*

$$W(P, P') \leqq AT(P, P'),$$

where A is the square root of the Poincaré area of U/G.

PROOF. We may assume that P is the origin of $T(G)$. Let $\nu=k\bar{\phi}/|\phi|$ be the unique Teichmüller differential such that $P'=P^\nu$. Let $\nu_1=\bar{\phi}/|\phi|$. Since $\phi(z)\not\equiv 0$, we have that $\nu_1\notin N(G)$. Let $\nu_2, \cdots, \nu_s$ be elements of $B(G)$ such that the equivalence classes of $\nu_1, \cdots, \nu_s$ form a basis of $B(G)/N(G)$, and let $\zeta(\mu)$ be the coordinate functions in a neighborhood N of P defined by this basis. Finally, let C denote the line $C: t\mapsto P^{t\nu}$, $t\in[0,1]$, and $L(C)$ its length in the Weil-Petersson metric. (C is the Teichmüller geodesic from P to P'.) We shall determine $L(C)$.

Assume first that $C\subset N$, so that

$$\zeta(t\nu) = (tk, 0, \cdots, 0). \tag{4}$$

Then it follows from (3) and (4) that

$$L(C) = \int_0^1 k\langle L^{t\nu}_{\nu_1}, L^{t\nu}_{\nu 1}\rangle^{1/2}\, dt.$$

It is easily seen from (1) that $|L^{t\nu}_{\nu_1}|=(1-t^2k^2)^{-1}$ so that from (2),

$$L(C) = \left\{\iint_{U/G^{t\nu}} (1-|z|^2)^{-2}\, dx\, dy\right\}^{1/2} \int_0^1 \frac{k}{1-t^2k^2}\, dt.$$

The left integral is the Poincaré area of the Riemann surface $U/G^{t\nu}$ which depends only on the isomorphism class of $G^{t\nu}$ and is therefore a constant for the Teichmüller space $T(G)$. Hence

$$W(P, P') \leqq L(C) = AT(P, P')$$

as was to be shown.

If the line C is not contained in N, we cover C by coordinate neighborhoods as follows. At each point P^{av} of C, $0 \leqq a \leqq 1$, consider the Beltrami differential $\rho \in B_1(G^{av})$ defined by $w^v = w^\rho \circ w^{av}$. It is easily shown that ρ is the unique Teichmüller differential whose image under the surjection $B_1(G^{av}) \to T(G^{av})$ is equal to P^v. ($T(G^{av}) = T(G)$ with origin P^{av}.) Set $\mu_1 = \rho/\|\rho\|_\infty$ and extend to a basis of $B(G^{av})/N(G^{av})$ as before. Let N^a denote the coordinate neighborhood of P^{av} with respect to this basis. Since C is compact, a finite number of the N^a cover C. Let these be denoted by $N_1, \cdots, N_n$, ordered by their origins, with $N_1 = N$. Let $P^{a_0 v}, P^{a_1 v}, \cdots, P^{a_n v}$, $0 = a_0 < a_1 < \cdots < a_n = 1$, be points of C such that $P^{a_j v} \in N_j \cap N_{j+1}$, $j = 1, \cdots, n-1$. If C_j is the segment with endpoints $P^{a_{j-1} v}$ and $P^{a_j v}$, $j = 1, \cdots, n$, then $C_j \subset N_j$. We repeat the previous argument to evaluate $L(C_j)$. (Slightly more work is involved due to the new coordinate system.) We obtain

$$L(C_j) = A\,\frac{1}{2}\left\{\log \frac{1 + a_{j+1}k}{1 - a_{j+1}k} - \log \frac{1 + a_j k}{1 - a_j k}\right\}.$$

Summing these to obtain $L(C)$, we again have the desired result.

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