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ON POLYNOMIALS SATISFYING A TURÁN TYPE INEQUALITY

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ABSTRACT. For Legendre polynomials $P_n(x)$, P. Turán has established the inequality

$$\Delta_n(x) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geqq 0, \qquad -1 \leqq x \leqq 1, n \geqq 1,$$

with equality only for $x=\pm 1$. This inequality has generated considerable interest, and analogous inequalities have been extended to various classes of polynomials: ultraspherical, Laguerre, Hermite, and a class of Jacobi polynomials. Our purpose here is to determine necessary and sufficient conditions for a general class of polynomials to satisfy a Turán type inequality and to characterize the generating functions of such a class.

1. **Introduction.** In 1948, Szegö [**12**] called attention to the following remarkable inequality of P. Turán for Legendre polynomials $P_n(x)$:

$$(1.1) \quad \Delta_n(x) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geqq 0, \qquad -1 \leqq x \leqq 1, n \geqq 1,$$

with equality only for $x=\pm 1$. This inequality has generated considerable interest (see, e.g., [**4**] and [**10**]). Turán's proof and three additional proofs of (1.1) were given by Szegö [**12**], who also extended the result to ultraspherical, Laguerre, and Hermite polynomials. More recently, Gasper [**2**] proved the analogue of (1.1) for a class of Jacobi polynomials. Our purpose here is to determine necessary and sufficient conditions for a general class of polynomials to satisfy a Turán type inequality and to characterize the generating functions of such a class.

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers with $a_0=1$, let

$$(1.2) \qquad g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k,$$

and let

$$\Delta_n(x) = g_n^2(x) - g_{n+1}(x)g_{n-1}(x), \qquad n \geqq 1.$$

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If, for each x, $-\infty < x < \infty$, either

$$\text{(T)} \qquad \Delta_n(x) > 0, \quad n \geqq 1, \qquad \text{or} \qquad \Delta_n(x) = 0, \quad n \geqq 1,$$

then we shall say that the sequence $\{g_n\}_{n=0}^{\infty}$ satisfies a *Turán type inequality.*

J. L. Burchnall [**1**] showed that if $g_n(x)$ has real, simple zeros for $n \geqq 1$, then $\{g_n\}$ satisfies condition (T). In addition, it is easy to see that if

$$g_n(x) = (1 + a_1 x)^n, \qquad n \geqq 1, \tag{1.3}$$

then $\Delta_n(x) \equiv 0$, $n \geqq 1$, so that the sequence $\{g_n\}$ defined by (1.3) trivially satisfies condition (T).

Now it is natural to inquire whether there are other examples of polynomials of the form (1.2) which satisfy a Turán type inequality. By way of an answer to this question, we shall show that provided the coefficients $\{a_k\}$ satisfy a mild restriction, the two sequences of polynomials mentioned above are the only sequences which satisfy a Turán type inequality, that is, satisfy condition (T). Indeed, if $\{g_n\}$ is a sequence of polynomials defined by (1.2), then we have

THEOREM 1. *If $\{g_n\}$ satisfies condition* (T) *and if $\Delta_n(\xi)=0$ for some $\xi \neq 0$, $n \geqq 1$, then*

$$g_n(x) = (1 + a_1 x)^n, \qquad n \geqq 1.$$

THEOREM 2. *If $\Delta_n(x) > 0$ for all $x \neq 0$, $n \geqq 1$, and if the sequence of coefficients $\{a_k\}$ satisfies the condition*

$$a_{k-1} a_{k+1} < 0 \quad \textit{whenever } a_k = 0, \tag{1.4}$$

then $g_n(x)$ has real, simple zeros for $n \geqq 1$.

We remark that it is not difficult to construct examples which show that Theorem 2 is false if condition (1.4) is omitted.

2. **Proof of Theorem 1.** If $\Delta_n(\xi)=0$ for some $\xi \neq 0$, $n \geqq 1$, then, in particular, $\Delta_1(\xi) = (a_1^2 - a_0 a_2)\xi^2 = 0$. Hence, $a_1^2 - a_0 a_2 = 0$ and *a fortiori* $\Delta_1(x) = (a_1^2 - a_0 a_2)x^2 \equiv 0$. But then, in view of condition (T),

$$\Delta_n(x) \equiv 0, \qquad n = 1, 2, \cdots. \tag{2.1}$$

Now $\Delta_n(x)$ is a polynomial of degree $2n$ with leading coefficient $a_n^2 - a_{n-1}a_{n+1}$, so (2.1) implies that

$$a_n^2 - a_{n-1} a_{n+1} = 0, \qquad n = 1, 2, \cdots. \tag{2.2}$$

Thus, it follows from (2.2) and an easy induction argument that $a_n = a_1^n$, $n = 1, 2, \cdots$. Hence,

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k = \sum_{k=0}^{n} \binom{n}{k} a_1^k x^k = (1 + a_1 x)^n.$$

3. **Proof of Theorem 2.** The proof of Theorem 2 depends upon an algebraic rule, which Pólya [**6**, p. 21] credits to de Gua, and a lemma.

de Gua's rule. A polynomial $f(x)$ with real coefficients has real, simple zeros only, if its derivatives $f'(x), f''(x), \cdots, f^{(n)}(x), \cdots$ have the property:

If ξ is real and $f^{(n)}(\xi)=0$, then $f^{(n-1)}(\xi)f^{(n+1)}(\xi)<0$.

LEMMA. *Under the hypothesis of Theorem* 2,

$$a_n^2 - a_{n-1}a_{n+1} > 0, \qquad n = 1, 2, \cdots.$$

PROOF. The proof will be by induction. First observe that, by hypothesis, $\Delta_1(x)=(a_1^2-a_0a_2)x^2>0$, $x\neq 0$, and hence,

$$a_1^2 - a_0a_2 > 0. \tag{3.1}$$

Now suppose

$$a_1^2 - a_0a_2 > 0, \quad a_2^2 - a_1a_3 > 0, \quad \cdots, \quad a_{n-1}^2 - a_{n-2}a_n > 0. \tag{3.2}$$

To show that $a_n^2-a_{n-1}a_{n+1}>0$, note that

$$\Delta_n(x) = \sum_{k=2}^{2n} c_kx^k,$$

where

$$c_{2n} = a_n^2 - a_{n-1}a_{n+1} \quad \text{and} \quad c_{2n-1} = (n-1)(a_na_{n-1} - a_{n-2}a_{n+1}).$$

Thus, the hypothesis $\Delta_n(x)>0$, $x\neq 0$, implies $a_n^2-a_{n-1}a_{n+1}\geqq 0$ and

$$a_na_{n-1} = a_{n-2}a_{n+1} \quad \text{whenever } a_n^2 - a_{n-1}a_{n+1} = 0. \tag{3.3}$$

Now if $a_n=0$, then it follows from (1.4) that $a_n^2-a_{n-1}a_{n+1}>0$. If, on the other hand, $a_n\neq 0$ and $a_n^2-a_{n-1}a_{n+1}=0$, then (3.3) implies $a_na_{n-1}=a_{n-2}a_{n+1}$. Consequently, it follows that $a_{n-1}^2-a_{n-2}a_n=0$. This contradicts (3.2) and thus, the induction is complete.

We now proceed with the proof of Theorem 2. First, set

$$P_n(x) = (1/n!)x^ng_n(x^{-1}) \tag{3.4}$$

and observe that

$$P_n'(x) = P_{n-1}(x). \tag{3.5}$$

Next, express $x^{2n}\Delta_n(x^{-1})$ in terms of the polynomials defined by (3.4) to obtain

$$x^{2n}\Delta_n(x^{-1}) = (n+1)!\,(n-1)!\left[\frac{n}{n+1}P_n^2(x) - P_{n-1}(x)P_{n+1}(x)\right]. \tag{3.6}$$

Since by hypothesis $\Delta_n(x)>0$ for $x\neq 0$, $n\geqq 1$, (3.6) implies

$$(3.7)\quad \sigma_n(x) = \frac{n}{n+1} P_n^2(x) - P_{n-1}(x)P_{n+1}(x) > 0, \qquad x \neq 0,\ n \geqq 1.$$

Moreover, the preceding lemma implies

$$(3.8)\quad (n-1)!\,(n+1)!\,\sigma_n(0) = a_n^2 - a_{n-1}a_{n+1} > 0, \qquad n \geqq 1.$$

Thus, by (3.7) and (3.8), we have

$$(3.9)\qquad \sigma_n(x) > 0, \qquad -\infty < x < \infty, \qquad n \geqq 1.$$

Now suppose that $P_n^{(k)}(\xi)=0$. Then by (3.5) and (3.9), we have

$$\begin{aligned} 0 < \sigma_{n-k}(\xi) &= \frac{n-k}{n-k+1} P_{n-k}^2(\xi) - P_{n-k+1}(\xi)P_{n-k-1}(\xi) \\ &= \frac{n-k}{n-k+1} [P_n^{(k)}(\xi)]^2 - P_n^{(k-1)}(\xi)P_n^{(k+1)}(\xi) \\ &= -P_n^{(k-1)}(\xi)P_n^{(k+1)}(\xi). \end{aligned}$$

Thus,

$$P_n^{(k-1)}(\xi)P_n^{(k+1)}(\xi) < 0, \qquad k = 1, \cdots, n-1,$$

and de Gua's rule implies $P_n(x)$ has real, simple zeros for $n\geqq 1$. Since $g_n(x)=n!\,x^{-n}P_n(x^{-1})$, it follows that $g_n(x)$ has real, simple zeros for $n\geqq 1$.

Theorem 2 has the following immediate but interesting

COROLLARY. *Let $\Delta_n(x)$ and $\{a_n\}$ satisfy the hypothesis of Theorem* 2 *and set*

$$g_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} a_{k+p}x^k, \qquad n \geqq 1,\ p \geqq 1.$$

Then $g_{n,p}(x)$ has real, simple zeros and, for every $p\geqq 1$, the sequence $\{g_{n,p}\}$ satisfies condition (T).

PROOF. Since

$$g_{n,p}(x) = \frac{p!}{(n+p)!} g_{n+p}^{(p)}(x)$$

and $g_{n+p}(x)$ has real, simple zeros, Rolle's theorem implies that $g_{n,p}(x)$ has real, simple zeros. The assertion that $\{g_{n,p}\}$ satisfies condition (T), for $p\geqq 1$, then follows from Burchnall's result mentioned in the Introduction.

4. The generating functions of polynomials satisfying condition (T). Suppose

$$f(z) = \sum_{k=0}^{\infty} a_k z^k/k! \qquad (a_0 = 1) \tag{4.1}$$

is holomorphic in a neighborhood of the origin. It is well known (see, e.g., [**8**]) that the sequence of polynomials $\{g_n\}$, defined by (1.2), is generated by $e^z f(xz)$, that is, $e^z f(xz) = \sum_{n=0}^{\infty} g_n(x) z^n/n!$, while the sequence of polynomials $\{P_n\}$, defined by (3.4), is generated by $e^{xz} f(z)$, that is, $e^{xz} f(z) = \sum_{n=0}^{\infty} P_n(x) z^n$. (The polynomials $P_n(x)$ are called *Appell polynomials.*)

Of special interest is the case when $f(z)$ is of the form

$$f(z) = e^{-\gamma z^2 + \beta z} \prod_{n=1}^{\infty} (1 - z/z_n) e^{z/z_n} \tag{4.2}$$

where $\gamma \geqq 0$, β, z_n are real and $\sum_{n=1}^{\infty} z_n^{-2} < \infty$.

We shall say that an entire function $f(z)$ of the form (4.2) belongs to the *class* $\mathscr{L}$-$\mathscr{P}$ (Laguerre-Pólya) and we shall write $f(z) \in \mathscr{L}$-$\mathscr{P}$.

If $f(z) \in \mathscr{L}$-$\mathscr{P}$ is given by (4.1), then it is well known [**7**, p. 110] that, for $n \geqq 1$, $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$ has only real zeros. Consequently, it follows ([**3**, §4.3] or [**9**, p. 76]) that

$$a_k^2 - a_{k-1}a_{k+1} > 0, \quad k \geqq 1, \quad \text{or} \quad a_k^2 - a_{k-1}a_{k+1} = 0, \quad k \geqq 1.^{1} \tag{4.3}$$

(Note that the second condition in (4.3) implies that $f(z) = e^{a_1 z}$.) Since $f(z) \in \mathscr{L}$-$\mathscr{P}$ clearly implies $e^z f(xz) \in \mathscr{L}$-$\mathscr{P}$ and $e^{xz} f(z) \in \mathscr{L}$-$\mathscr{P}$ for every x, $-\infty < x < \infty$, the following proposition is a consequence of (4.3).

PROPOSITION 1. *Let $f(z)$ be given by* (4.1). *If $f(z) \in \mathscr{L}$-$\mathscr{P}$, then the polynomial sequences $\{g_n\}$ and $\{n!\,P_n\}$ generated by $e^z f(xz)$ and $e^{xz} f(z)$ respectively, satisfy condition* (T).

Conversely, as a consequence of Theorem 2, we have

PROPOSITION 2. *If $\{a_k\}_{k=0}^{\infty}$, $a_0 = 1$, is a sequence of real numbers which satisfies* (1.4) *and if the sequence $\{g_n\}$ defined by* (1.2) *satisfies condition* (T) *then the function*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k/k!$$

belongs to the class $\mathscr{L}$-$\mathscr{P}$.

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[1] Szegö [**12**] used this condition to show that many of the classical polynomials satisfy a Turán type inequality.

PROOF. By Theorems 1 and 2, $g_n(x)$, $n \geqq 1$, has only real zeros, and hence, the polynomial $G_n(z)$, $n \geqq 1$, where

$$G_n(z) = g_n\left(\frac{z}{n}\right) = \sum_{k=0}^{n} \frac{a_k}{k!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) z^k$$

has only real zeros. Furthermore, since

$$G_n(0) = 1, \quad |G_n'(0)| = |a_1| \quad \text{and} \quad |G_n''(0)| \leqq \tfrac{1}{2}\,|a_2|,$$

it follows (see, e.g., Szász [11]) that $\{G_n(z)\}$ is a normal family. Now $f(z)$ is clearly the unique limit function of the sequence $\{G_n(z)\}$; thus, $\{G_n(z)\}$ converges uniformly to $f(z)$ on every compact subset of the plane. Since $G_n(z)$, $n \geqq 1$, has only real zeros, a classical result of Pólya [5] implies that $f(z) \in \mathscr{L}\text{-}\mathscr{P}$.

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