ON ISOMORPHIC GROUPS AND HOMEOMORPHIC SPACES

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Abstract. Let $C(X, G)$ denote the group of continuous functions from a topological space $X$ into a topological group $G$ with the pointwise multiplication. Some classes of $SQ$-pairs and properties of the corresponding topological group $C(X, G)$ with the compact-open topology are investigated. We also show that the existence of a group isomorphism between groups $C(X, G)$ and $C(Y, G)$ implies the existence of a homeomorphism between $X$ and $Y$, if $(X, G)$ and $(Y, G)$ are $SQ$-pairs.

1. Introduction. For a topological space $X$ and a topological group $G$, let $C(X, G)$ be the group of all continuous functions from $X$ into $G$ with the pointwise multiplication, that is, $(fg)(x) = f(x)g(x)$; the identity element of the group $C(X, G)$ is the constant function $I_0(X, G)$, or simply $I_0$, which maps every $x$ in $X$ into the identity element $e$ of $G$. It is well known that if $C(X, G)$ is endowed with the compact-open topology, it becomes a topological group. It is clear that if $h$ is a homeomorphism of $X$ onto $Y$, then $f \mapsto f \circ h$ is an isomorphism from $C(Y, G)$ onto $C(X, G)$ which maps every constant function on $Y$ into the corresponding constant function on $X$. We are concerned, in this paper, with the question: If a group isomorphism exists between $C(Y, G)$ and $C(X, G)$ which maps every constant function on $Y$ into the corresponding constant function on $X$, does there exist a homeomorphism between $X$ and $Y$? In general, the answer to this question is, of course, no, for we may take $X$ to be a noncompact pseudocompact space, and then there is a ring isomorphism between the rings $C(X, R)$ and $C(\beta X, R)$ but $X$ and $\beta X$ are not homeomorphic.

We find that the answer to the above question is yes for certain pairs $(X, G)$ of topological space $X$ and topological group $G$. Such pairs are termed $SQ$-pairs as defined in [9]. §3 is devoted to proving this assertion by showing first that, if $X$ is a $k$-space, $X$ is homeomorphic to the space of all $c$-continuous homomorphisms of the topological group $C(X, G)$ onto the topological group $G$ with $F$-normal subgroups as kernels and...
endowed with the compact-open topology. We disclose some classes of SΩ-pairs, and some properties of \( C(X, G) \) in §2.

All topological spaces considered here are assumed to be Hausdorff.

2. SΩ-pairs. For each \( p \in X \), let \( M_p = \{ f \in C(X, G) : f(p) = e \} \), and let \( h_p \) be the evaluation map of \( C(X, G) \) onto \( G \) defined by \( h_p(f) = f(p) \).

For each \( r \in G \), let \( r \) denote the constant function in \( C(X, G) \) which maps \( X \) into \( r \). Then \( h_p \) is a continuous homomorphism of \( C(X, G) \) onto \( G \) with \( M_p \) as its kernel and maps every constant function \( r \) into \( r \). Hence we see that \( C(X, G)/M_p \) is isomorphic to \( G \) under the continuous isomorphism that maps every coset \( eM_p \) into \( e \). Note that for each \( p \) in \( X \), every coset \( eM_p \) contains exactly one constant map, namely \( e \). For the sake of convenience, let us call a homomorphism of \( C(X, G) \) (or \( C(X, G)/M \) onto \( G \)) a \( c \)-homomorphism if it maps every \( r \) (resp. \( rM \)) into \( r \). Every evaluation map is a \( c \)-continuous homomorphism of \( C(X, G) \) onto \( G \).

In contrast to the fact that every nonzero homomorphism of \( C(X) = C(X, R) \) onto \( R \) is a \( c \)-homomorphism [5, 10.5], not every continuous homomorphism of \( C(X, G) \) onto \( G \) is a \( c \)-continuous homomorphism, as the following example shows.

Example. Let \( G \) be the additive group of integers modulo 2 with the discrete topology. Then \( C(G, G) = \{ 0, f_1, f_2, f_3 \} \), where \( f_1 \) is the function which maps \( G \) into 1, \( f_2 \) is the function which maps 1 into 1 and 0 into 0, and \( f_3 \) is the one which maps 0 into 1 and 1 into 0. The compact-open topology for \( C(G, G) \) is the discrete topology. If we define a mapping \( h : C(G, G) \rightarrow G \) by defining \( h(0) = h(f_1) = 0 \), \( h(f_2) = h(f_3) = 1 \), then \( h \) is an onto homomorphism, yet it is not a \( c \)-homomorphism.

For \( f \in C(X, G) \), we let \( Z(f) = \{ x \in X : f(x) = e \} \), and for a subgroup \( M \) of \( C(X, G) \), let \( Z(M) = \{ Z(f) : f \in M \} \). Note that, for any \( f \) and \( g \) in \( C(X, G) \),

\[ Z(fg) \supset Z(f) \cap Z(g), \quad Z(f^{-1}) = Z(f) \quad \text{and} \quad Z(fgf^{-1}) = Z(g). \]

Definition 1 [9]. We shall call a pair \((X, G)\) of a topological space \( X \) and a topological group \( G \) an \( S \)-pair if, for each closed subset \( C \) of \( X \) and \( x \notin C \), there exists \( f \in C(X, G) \) such that \( Z(f) \supset C \) and \( f(x) \neq e \).

It is clear that \((X, R)\) is an \( S \)-pair for every completely regular space, and that if \((X, G)\) is an \( S \)-pair then \( X \) is completely regular.

Remark 1. If \( X \) is a topological space such that each \( x \) in \( X \) has a local base \( U_x \) satisfying the property that, for each \( U \) in \( U_x \) there exists a continuous function \( f \) of \( U \) into \( G \) such that \( f(x) \neq e \) but \( f(y) = e \) for each \( y \) in \( U - U \), then \((X, G)\) is an \( S \)-pair. To see this, let \( C \) be a closed subset of \( X \) and \( x \notin C \). Then, for some \( U \) in \( U_x \), \( x \in U \subset X - C \); and let \( f \) be a continuous function on \( U \) into \( G \) such that \( f(x) \neq e \) but \( f(y) = e \) for each \( y \)
in $\bar{U} = U$. Define $g: X \rightarrow G$ such that $g = f$ on $\bar{U}$ and $g(y) = e$ for $y \notin \bar{U}$. Then $g \in C(X, G)$, $Z(g) \supseteq C$, and $g(x) \neq e$.

**Remark 2.** If $X$ is completely regular, and $G$ is path connected, then $(X, G)$ is an $S$-pair. To see this let $t \neq e$ be in $G$. If $C$ is a closed subset of $X$ and $x \notin C$, let $f$ be a continuous function of $X$ into $[0, 1]$ such that $f(x) = 1$ and $f(C) = \{0\}$, and let $g: [0, 1] \rightarrow G$ be the path such that $g(0) = e$ and $g(1) = t$. Then $g \circ f$ is the desired function in $C(X, G)$.

**Remark 3.** For every zero-dimensional space $X$, $(X, G)$ is an $S'$-pair. We point out that, if $B$ is a closed subset of $X$ and $(X, G)$ is an $S$-pair, then $(B, G)$ is also an $S$-pair.

**Definition 2 [9].** (1) A normal subgroup $M$ of $C(X, G)$ is called an $F$-normal subgroup if $\{Z(f) : f \in M\}$ has the finite intersection property.

(2) A pair $(X, G)$ of a topological space $X$ and a topological group $G$ is called a $Q$-pair if whenever $M$ is an $F$-normal subgroup of $C(X, G)$ such that $C(X, G)/M$ is isomorphic to $G$ by a $c$-isomorphism, then $\bigcap Z(M) \neq \emptyset$.

It is clear that if $X$ is a completely regular space such that $(X, R)$ is a $Q$-pair, then $X$ is realcompact. As pointed out in [9], $(X, G)$ is a $Q$-pair if $X$ can be embedded into $G$ as a subspace of $G$. Since every completely regular space $X$ is a closed subspace of the free topological group $F(X)$ generated by $X$, and every topological group can be embedded as a closed subgroup of a path connected and locally path connected topological group [6], we see that for every completely regular space $X$ there exists a path connected and locally path connected topological group $G$ such that $(X, G)$ is an $S'_Q$-pair. If $X$ is compact, $(X, R)$ is an $S'_Q$-pair.

If $(X, G)$ is a $Q$-pair, then the only $F$-normal subgroups of $C(X, G)$ such that $C(X, G)/M$ is $c$-isomorphic to $G$ are of the form $M_p$, $p \in X$ [9]. Thus we have the following:

**Proposition 4.** An $S$-pair $(X, G)$ is a $Q$-pair if and only if every $c$-homomorphism $h$ of $C(X, G)$ onto $G$ with an $F$-normal subgroup as its kernel is of the form $h_p$ for some $p \in X$.

**Proof.** For the necessity, let $M$ be the kernel of $h$, then $C(X, G)/M$ is $c$-isomorphic to $G$. Hence there is $p \in \bigcap Z(M)$ such that $M = M_p$. Therefore $\ker h = \ker h_p$. Now for $f \in C(X, G)$, let $f(p) = c$, and let $g = fe^{-1}$, then $g \in M_p = M$. Hence

$$h(f) = h(gc) = h(g)h(c) = h(g)c = c = f(p) = h_p(f).$$

This shows that $h = h_p$.

For the sufficiency, suppose $M$ is an $F$-normal subgroup of $C(X, G)$ such that $C(X, G)/M$ is $c$-isomorphic to $G$ by the $c$-isomorphism $k$. Let $h = k \circ \alpha$, where $\alpha$ is the natural map of $C(X, G)$ onto $C(X, G)/M$. Then $h$...
is a $c$-homomorphism of $C(X, G)$ onto $G$ with $M$ as its kernel. Hence there is a unique $p \in X$ such that $h = h_p$, and thus $M = M_p$.

Following [7], we call a topological space $X$ a $V$-space if for points $p, q, x, y$ of $X$, where $p \neq q$, there exists a continuous function $f$ of $X$ into itself such that $f(p) = x$ and $f(q) = y$. It is shown in [7] that every completely regular path connected space and every zero-dimensional space is a $V$-space.

Recall that a topological space $X$ is said to be an $S$-space if, for each pair of distinct points of $X$, there is a continuous real-valued function on $X$ whose values at these points do not coincide. R. Arens defined it in [1], and has shown that, if the space $C(X, R)$ satisfies the first axiom of countability and $X$ is an $S$-space, then $X$ is hemicompact. Adopting the same line of argument, we have the following:

**Theorem 5.** If $(X, G)$ is an $S$-pair, $G$ is a $V$-space, and if $C(X, G)$ satisfies the first axiom of countability, then $X$ is hemicompact and $G$ is metrizable.

**Proof.** Since $G$ can be embedded as a retract of $C(X, G)$, $G$ is metrizable. For the hemicompactness of $X$, the proof is not different from that of [1, Theorem 8] and thus omitted.

It is remarked that, if $X = \bigcup_{n=1}^{\infty} C_n$ where $C_1 \subseteq C_2 \subseteq C_3, \ldots$, is hemicompact and if $\{V_n\}$ is a countable local base for $e$ in $G$, then $\{(C_n, V_m)\}$ is a local base at $I_0$ in $C(X, G)$, and hence $C(X, G)$ is metrizable, where $(C_n, V_m) = \{f \in C(X, G) : f(C_n) \subseteq V_m\}$.

**Lemma 6.** Let $(X, G)$ be an $S$-pair, and let $\Omega$ be an open covering for $X$. For each closed subset $C$ of $X$ contained in some member of $\Omega$ and for each open neighborhood $U$ of $e$ in $G$, let $(C, U) = \{f \in C(X, G) : f(C) \subseteq U\}$. Then the topology $t$ for the group $C(X, G)$ having the family of sets of the form $(C, U)$ as subbasic neighborhoods of $I_0$ is jointly continuous, that is, the map $P : X \times C(X, G) \to G$ defined by $P(f, x) = f(x)$ is continuous.

**Proof.** Let $f \in C(X, G)$, $x \in X$, and let $W$ be a neighborhood of $f(x)$. Then $f(x)U \subseteq W$ for some open set $U$ in $G$ containing $e$, and hence $x \in f^{-1}(f(x)V) \cap O$, where $x \in O \in \Omega$ and $V$ an open neighborhood of $e$ such that $V^2 \subseteq U$. If $C$ is a closed neighborhood of $x$ such that $C \subseteq f^{-1}(f(x)V) \cap O$, then, for $g \in f(C, V)$ and $y \in C$, $g(y) \in f(y)V \subseteq f(x)U \subseteq W$. Hence $P$ is continuous.

**Theorem 7.** Let $(X, G)$ be an $S$-pair, where $G$ is a $V$-space. If there exists a smallest jointly continuous topology $t$ for the group $C(X, G)$, then $X$ is locally compact.
Proof. The proof is similar to that of [1, Theorem 3]. Let \( a \) be an element of \( G \) different from \( e \), and let \( U \) be a neighborhood of \( e \) in \( G \) such that \( a \notin U \), and let \( x \in X \). By the joint continuity of \( t \), let \( V \) be a neighborhood of \( x \), and \( W \) a \( t \)-neighborhood of \( I_0 \) such that \( g(V) \subset U \) for every \( g \) in \( W \). We want to show that \( \mathcal{P} \) is compact.

Let \( \Omega \) be an open covering for \( \mathcal{P} \), and let \( \Omega' = \{X - \mathcal{P}\} \cup \Omega \). Then \( \Omega' \) is an open covering for \( X \). Let \( t' \) be the topology for \( C(X, G) \) induced by \( \Omega' \) as described in Lemma 6, then we have \( t \subset t' \). Hence there are closed sets \( C_i \subset O_i \) of \( X \) and open neighborhoods \( U_i \) of \( e \) in \( G \), \( i = 1, 2, \ldots, n \), such that \( W' = \bigcap_{i=1}^n (C_i, U_i) \) is contained in \( W \). Let \( O = V - \bigcup_{i=1}^n C_i \), and suppose that \( p \in O \). Then there is \( f \) in \( C(X, G) \) such that \( Z(f) \supset X - O \) and \( f(p) \neq e \). Let \( g \) be a continuous function of \( G \) into itself with \( g(e) = e \) and \( g(f(p)) = a \), and let \( g = g \circ f \). Then \( h(X - O) = e \) and \( h(p) = a \notin U \), hence \( h \in W' \). But \( p \) is in \( V \) and \( h(p) = a \notin U \); we have \( h \notin W \) which is impossible. Hence \( O = \emptyset \), and we have \( \mathcal{P} \subset \bigcup_{i=1}^n C_i \subset \bigcup_{i=1}^n O_i \). Therefore \( \mathcal{P} \) is compact.

Corollary. If \( (X, G) \) is an S-pair, where \( G \) is a V-space, and \( X \times C(X, G) \) is a k-space, where \( C(X, G) \) has the compact-open topology, then \( X \) is locally compact.

Proof. If \( X \times C(X, G) \) is a k-space, then the compact-open topology for \( C(X, G) \) is jointly continuous [2]; hence \( X \) is locally compact.

The above corollary generalizes a result in [2]. As an application, we show in the following example that the product of two topological groups which are k-spaces need not be a k-space, a fact pointed out by N. Noble [8].

Example. Let \( X \) be the dual space of an infinite-dimensional Fréchet space with the compact-open topology. Then \( X \) is a topological group which is a hemicompact k-space but is not locally compact. If \( G \) is any metrizable topological group which is also a V-space such that \( (X, G) \) is an S-pair, then \( C(X, G) \) is metrizable by the remark following Theorem 5. Since \( X \) is not locally compact, \( X \times C(X, G) \) is a topological group but is not a k-space as follows from the above corollary. This example was cited by N. Noble [8] for the case where \( G \) is the additive group of real numbers.

3. Isomorphic groups. This section is devoted to prove the following:

**Theorem 8.** Suppose that \( (X, G) \) and \( (Y, G) \) are SQ-pairs. If there exists an isomorphism between groups \( C(Y, G) \) and \( C(X, G) \) which maps every constant function on \( Y \) into the corresponding constant function on \( X \), then \( X \) and \( Y \) are homeomorphic.

All pairs \( (Z, G) \) considered in this section are assumed to be SQ-pairs. Since every noncompact pseudocompact space \( X \) is not realcompact,
(X, R) cannot be a Q-pair, thus Theorem 8 is false if (X, G) is not a Q-pair.

In order to establish Theorem 8, we first prove that, if X is a k-space, X is homeomorphic to the space of all c-continuous homomorphisms of the topological group C(X, G) onto the topological group G with F-normal subgroups as kernels and endowed with the compact-open topology; let H(X, G) denote such a space of c-continuous homomorphisms. For each p ∈ X, the evaluation map h_p is in H(X, G), hence the correspondence p → h_p defines a map μ from X into H(X, G).

**Theorem 9.** If X is a k-space, the mapping μ is a homeomorphism of X onto H(X, G).

**Proof.** Proposition 4 implies that μ is onto.

If p ≠ q in X, there is f ∈ C(X, G) such that f(p) ≠ f(q), hence h_p(f) ≠ h_q(f). Thus μ is one-to-one.

The continuity of μ follows from Theorem 2 of [4], which states that if F is a family of continuous functions from a k-space X into a regular space Y endowed with the compact-open topology, then the mapping \( \theta : X \rightarrow C(F, Y) \) defined by \( \theta(x)(f) = f(x) \) is continuous, where \( C(F, Y) \) also has the compact-open topology.

It remains to show that μ is a closed map. Let C be a closed subset of X. Then μ(C) = \{ h_x : x ∈ C \}. Let \{ h_{x_n} \}_{n ∈ A} be a net in μ(C) such that h_{x_n} → h_x in H(X, G), where x_n ∈ C for each n ∈ A. If x ∉ C, then there exists an f in C(X, G) such that f(x) ∉ cl[f(C)]. But h_{x_n}(f) → h_x(f) in G; we have f(x_n) → f(x) in G, hence f(x) ∉ cl[f(C)], a contradiction. Hence x ∈ C and μ(C) is closed.

**Remark 10.** The hypothesis that X is a k-space in Theorem 9 is merely to assure the continuity of μ. In fact, if H(X, G) is given the point-open topology instead of the compact-open topology, the mapping μ is easily seen to be continuous without assuming that X is a k-space.

Suppose now that \( \theta : X \rightarrow Y \) is a continuous map of a k-space X into a k-space Y. Define \( \theta' : C(Y, G) \rightarrow C(X, G) \) by setting \( \theta'(g) = g \circ \theta \) for each g in C(Y, G) into the corresponding constant function in C(X, G). Note that if \( h_x \in H(X, G) \), then \( h_x \circ \theta' \) is in H(Y, G). Hence we have a continuous mapping \( \theta'' \) of H(X, G) onto H(Y, G) defined by \( \theta''(h_x) = h_x \circ \theta' \) for each \( h_x \in H(X, G) \). It is easy to verify that the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\theta} & Y \\
\downarrow n_x & & \downarrow \mu_Y \\
H(X, G) & \xrightarrow{\theta''} & H(Y, G)
\end{array}
\]

is commutative, where \( \mu_Z : Z \rightarrow H(Z, G) \) is the mapping of Theorem 9.
Theorem 11. Suppose that $X$ and $Y$ are $k$-spaces. Every continuous homomorphism $h: C(Y, G) \rightarrow C(X, G)$ which maps every constant function on $Y$ into the corresponding constant function on $X$, induces a unique continuous mapping $j$ of $X$ into $Y$ such that $j' = h$. Furthermore, if $h$ is a topological isomorphism, then the induced mapping $j$ is a homeomorphism.

Proof. Let $h'$ be the mapping of $H(X, G)$ into $H(Y, G)$ defined by $h'(h_x) = h_x\circ h$ for each $h_x \in H(X, G)$. Since $X$ and $Y$ are $k$-spaces, $\mu_X$ and $\mu_Y$ are homeomorphisms by Theorem 9. If we define $j: X \rightarrow Y$ by setting $j = \mu_Y^{-1} \circ h' \circ \mu_X$, then the above diagram shows that $j$ is continuous. Note that $j(x) = y$ if and only if $h(g)(x) = g(y)$ for each $g \in C(Y, G)$. If $j': C(Y, G) \rightarrow C(X, G)$ is the mapping defined by $j'(g) = g \circ j$ for each $g \in C(Y, G)$, it is easy to verify that $j' = h$.

If $r: X \rightarrow Y$ is any continuous mapping such that $r(x) \neq j(x)$ for some $x \in X$, then there exists an $f \in C(X, G)$ such that $f(r(x)) \neq f(j(x))$. Hence $r' \neq j'$, and the uniqueness of $j$ follows.

Now if $h$ is a topological isomorphism, then $j$ is onto and one-to-one (cf. [5, 10.2]), and $j^{-1}$ is continuous. Hence $j$ is a homeomorphism of $X$ onto $Y$, and the proof is completed.

As one may notice from the above proof, the introduction of the mapping $j$ depends solely on the homeomorphism of the maps $\mu_X$ and $\mu_Y$, and, as noted in Remark 10, the mapping $\mu$ is always a homeomorphism if $H(X, G)$ is endowed with the point-open topology which indeed coincides with the compact-open topology if the domain space is discrete [3]. With this remark, we can now prove Theorem 8 very easily; take discrete topologies for the groups $C(Y, G)$ and $C(X, G)$ then apply the proof of Theorem 11.

Remark 12. In fact, if we define an $S$-pair $(X, G)$ in a weaker form, (that is if we define $(X, G)$ to be an $S$-pair if, for each closed subset $C$ of $X$ and $x \notin C$ there exists an $f \in C(X, G)$ such that $f(x) \notin \text{cl}([f(C)])$, then most of the results stated above, except perhaps Theorems 5 and 7, hold.

References


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