WALLMAN-TYPE COMPACTIFICATIONS
ON 0-DIMENSIONAL SPACES
LI PI SU

Abstract. Let $E$ be Hausdorff 0-dimensional, $\mathcal{D}$ the discrete
space $\{0, 1\}$, and $\mathcal{N}$ the discrete space of all nonnegative integers.
Every $E$-completely regular space $X$ has a clopen normal base $\mathcal{F}$
with $X \setminus F \in \mathcal{F}$ for each $F \in \mathcal{F}$. The Wallman compactification
$\omega(\mathcal{F})$ is $\mathcal{D}$-compact. Moreover, if an $E$-completely regular space
$X$ has a countably productive clopen normal base $\mathcal{F}$ with $X \setminus F \in \mathcal{F}$
for each $F \in \mathcal{F}$, then the Wallman space $\eta(\mathcal{F})$ is $\mathcal{N}$-compact.
Hence, if $X$ has such an $\mathcal{F}$, and is an $\mathcal{F}$-realcompact space, then $X$
is $\mathcal{N}$-compact.

Recently, the relations between Stone-Čech compactifications and
Wallman compactifications, those between realcompactifications and
Wallman compactifications and those between $E$-compactifications and
Wallman compactifications have been studied by Frink [6], Njastad
[9], the Steiners [11], [12], Alo and Shapiro [1], [2], [3], [4], Piacun and
Su [10], and some others.

A topological space is said to be 0-dimensional if it has a base con-
sisting of clopen (both closed and open) subsets of $X$. For other notations
and terminology see one of [1], [2], [3], [4], [10], [11] and [12], and
[8].

Let $\mathcal{H}$ be a base for closed subsets of $E$. Let $X$ be a $T_\emptyset$-space. Let
$E(\mathcal{H})$ be the family of all subsets of $X$ of the form $f^{-1}[B]$ where for some
positive integer $n$, $f \in C(X, E^n)$ and $B \in \mathcal{H}$. According to the definition
of $E$-complete regularity (see [8]), $X$ is $E$-completely regular iff $E(\mathcal{H})$ is
a base for the closed subsets of $X$.

From now on we will let $E$ be a $T_2$ 0-dimensional space with card $E \geq 2$.
According to [8], the following theorems are true:

Theorem (Mrówka). The following three statements are equivalent:
(1) $X$ is a 0-dimensional $T_\emptyset$-space.
(2) $X$ is $E$-completely regular.
(3) $X$ is $\mathcal{D}$-completely regular.

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of some passages.
Theorem (Mrówka). Let $X$ be a compact, $0$-dimensional $T_0$-space. Then $X$ is $\mathcal{D}$-compact and $X$ is $E$-compact.

If $\mathcal{F}$ is a family of clopen subsets of $X$ such that $X \setminus F \in \mathcal{F}$ for each $F \in \mathcal{F}$, then $\mathcal{F}$ is a base for the open sets of $X$ iff $\mathcal{F}$ is a base for the closed sets of $X$. In the sequel we shall mainly be concerned with bases which are rings (closed under the operations of taking finite unions and finite intersections). We therefore make the following definition: $\mathcal{F}$ is called a complemental base on $X$ iff all of the following are satisfied

1. $\mathcal{F}$ is a family of clopen subsets of $X$.
2. $X \setminus F \in \mathcal{F}$ for each $F \in \mathcal{F}$.
3. $\mathcal{F}$ is a ring.
4. $\mathcal{F}$ is a base for closed sets of $X$.

It is obvious that any complemental base is a normal base. Also, if $X$ is $E$-completely regular and if $\mathcal{F}$ is a complemental base on $E$, then $\mathcal{F}(\mathcal{F})$ is a complemental base on $X$. Since the family of all clopen subsets of $E$ is a complemental base, it follows that every $E$-completely regular space $X$ has at least one complemental base. Conversely, if a $T_0$-space $X$ has a complemental base, then $X$ is necessarily $0$-dimensional and so, by Mrówka's Theorem quoted above, $X$ is $E$-completely regular.

In order to fix the notation, we repeat the construction of the Wallman spaces $\omega(\mathcal{F})$ and $\eta(\mathcal{F})$ which arise from a normal base $\mathcal{F}$. Thus, let $\mathcal{F}$ be a normal base on $X$. Let $\omega(\mathcal{F})$ be the set of all $\mathcal{F}$-ultrafilters and $\eta(\mathcal{F})$ the set of all $\mathcal{F}$-ultrafilters with the c.i.p. (countable intersection property). We now topologize $\omega(\mathcal{F})$ and $\eta(\mathcal{F})$ as follows: In $\omega(\mathcal{F})$, for each $F \in \mathcal{F}$ we define the set $F^* = \{ \emptyset \in \omega(\mathcal{F}) : F \subseteq \emptyset \}$. Then the set $\{ F^* : F \in \mathcal{F} \}$ can be taken as a base for closed subsets of $\omega(\mathcal{F})$. Similarly, in $\eta(\mathcal{F})$, we define $F^{**} = \{ F^* \in \eta(\mathcal{F}) : F \subseteq F^* \}$ for each $F \in \mathcal{F}$. $\omega(\mathcal{F})$ and $\eta(\mathcal{F})$ with the described topologies are called Wallman spaces. In fact $\omega(\mathcal{F})$ is a Hausdorff compactification of $X$. (See [6], [11].) Let $\phi$ be the natural embedding of $X$ into $\omega(\mathcal{F})$ (or $\eta(\mathcal{F})$) defined by identifying $\phi(x)$ with the $\mathcal{F}$-ultrafilter consisting of all $F \in \mathcal{F}$ that contain $x$, denoted by $\mathcal{O}_x = \{ F \in \mathcal{F} : x \in F \}$. It is clear $\mathcal{O}_x \in \omega(\mathcal{F})$ and $\eta(\mathcal{F})$.

The following Lemmas A, B and C are easy to prove.

Lemma A. Let $\mathcal{F}$ be a normal base on $X$. Then:
(a) $(F_1 \cap F_2)^* = F_1^* \cap F_2^*$, $(F_1 \cap F_2)^{**} = F_1^{**} \cap F_2^{**})$ for all $F_1, F_2 \in \mathcal{F}$.
(b) $\phi(F) = \phi(X) \cap F^*$ for all $F \in \mathcal{F}$.
(c) $\cl_{\omega(\mathcal{F})} \phi(F) = F^*$ for all $F \in \mathcal{F}$.

The proof is similar to that of Lemma I in [1].

Since $\mathcal{F}$ is a disjunctive family (see [1]) and $X$ is $T_1$, the mapping $\phi$ is a one-one mapping of $X$ onto the subspace $\phi(X)$ of $\omega(\mathcal{F})$ ($\eta(\mathcal{F})$).
According to Lemma A(b), $\phi$ is both continuous and closed. Hence $\phi$ is a homeomorphism.

**Theorem A.** Let $X$ be $E$-completely regular and let $F$ be a complemental base on $X$. Then $F^* = \{ F^*: F \in F \}$ ($F^{**} = \{ F^{**}: F \in F \}$) is a complemental base on $\omega(F)$ ($\eta(F)$ resp.). Hence $\omega(F)$ ($\eta(F)$) is $E$-completely regular.

**Proof.** Observe that for each $F \in F$, $\emptyset \in \omega(F) \setminus F^*$ iff $F \notin \emptyset$ iff $(X \setminus F)^* \in \emptyset$ iff $\emptyset \in (X \setminus F)^*$. Therefore, for each $F \in F$, $\omega(F) \setminus F^* = (X \setminus F)^*$. It follows that $F^*$ is a complemental base on $\omega(F)$. Similarly, $F^{**}$ is a complemental base on $\eta(F)$. Since $\omega(F)$ and $\eta(F)$ are Hausdorff spaces, the theorem is true.

**Lemma B.** In addition to the conditions in Lemma A, if $F$ is countably productive, then:

(a) For $F_n \in F$, $n = 1, 2, \cdots$, $(\bigcup_{n=1}^{\infty} F_n)^{**} = \bigcup_{n=1}^{\infty} F^{**}$ and

$$\left( \bigcap_{n=1}^{\infty} F_n \right)^{**} = \bigcap_{n=1}^{\infty} F_n^{**}.$$  

(b) If $\mathcal{A}$ is a $F^*$- ($F^{**}$-) ultrafilter (with the c.i.p.) then $\emptyset = \{ F: F^* \in \mathcal{A} \}$ ($\emptyset^{**} = \{ F: F^{**} \in \emptyset \}$) is an $F$-ultrafilter (with the c.i.p.). And conversely.

The proof is similar to that of Theorem 1 in [4].

**Theorem B.** Let $X$ be an $E$-completely regular space and let $F$ be a complemental base on $X$. Then the Wallman compactification $\omega(F)$ is $E$-compact and also is $\mathcal{D}$-compact.

**Proof.** Theorem A implies that $\omega(F)$ is $E$-completely regular. Now apply Mrówka’s theorems quoted above using the known fact that $\omega(F)$ is a compact Hausdorff space.

In general, we do not know that if the Wallman spaces $\omega(F')$, $\eta(F')$ of an $E$-completely regular space generated by the ring $F'$ of all $E$-closed subsets of $X$ is $E$-completely regular. However according to Theorem A, we have

**Corollary A.** If $X$ is an $E$-completely regular space and if $F$ is a complemental base on $X$, then the Wallman spaces $\omega(F)$ and $\eta(F)$ are $E$-completely regular.

If, in particular, $E$ is either $\mathcal{N}$, the discrete space of the nonnegative integers, or $\mathcal{D}$, the discrete space $\{0, 1\}$. An $E$-closed subset of $X$ is a subset $A$ of $X$ such that there is a positive integer $n$ and a continuous function $f \in C(X, E^n)$ such that $A = f^{-1}[F]$ for some closed subset $F$.
of $E^n$. Since $F$ is clopen in $E^n$, each $E$-closed subset $A$ of $X$, and $X \setminus A$ are indeed $E$-clopen sets. Let $\mathcal{F}_1$ denote the family of all such $E$-clopen sets. Then, by [8, (3.18)] $\mathcal{F}_1$ is a ring. It is easy to show that $\mathcal{F}_1$ is a complemental base on $X$ iff $X$ is $E$-completely regular.

**Corollary B.** Let $E$ be $\mathcal{N}$ or $\mathfrak{D}$. For any $E$-completely regular space $X$, the Wallman spaces $\omega(\mathcal{F}_1)$, $\eta(\mathcal{F}_1)$ arising out of the ring $\mathcal{F}_1$ of $E$-closed (indeed it is $E$-clopen) subsets of $X$ is $E$-completely regular.

In a recent paper Chew [5] has given the following characterization of $\mathcal{N}$-compactness. We recall it here.

**Theorem C.** In a 0-dimensional space $X$, the following are equivalent:

(i) $X$ is $\mathcal{N}$-compact.

(ii) Every clopen ultrafilter on $X$ with the c.i.p. is fixed, (i.e., has non-empty intersection).

(iii) The collection of all the countable clopen coverings of $X$ is complete.

According to Frolik [7], let $\alpha=\{\mathcal{U}\}$ be a collection of coverings of a space $X$. An $\alpha$-Cauchy family $\mathcal{G}$ is a filter of subsets of $X$ such that for every $\mathcal{U} \in \alpha$, there exist $U$ in $\mathcal{U}$ and $G$ in $\mathcal{G}$ with $U \supseteq G$. The collection $\alpha$ is complete iff $\mathcal{G}$ is fixed (i.e., $\bigcap \mathcal{G} \neq \emptyset$) for each $\alpha$-Cauchy family $\mathcal{G}$.

The following lemmas are needed to show that there is a Wallman space $\eta(\mathcal{F})$ that is $\mathcal{N}$-compact.

**Lemma C.** Let $X$ be $E$-completely regular and let $\mathcal{F}$ be a complemental base on $X$ which is countably productive. Then every $\mathcal{F}$**-ultrafilter with the c.i.p. is fixed.

Proof. is straightforward from Lemma B(b).

**Lemma D.** Let $\mathcal{B}$ be a base consisting of clopen subsets of $X$. If the collection $\beta$ of all countable coverings from $\mathcal{B}$ is complete, then the collection $\alpha$ of all countable clopen coverings is complete.

**Proof.** Let $\mathcal{A}$ be an $\alpha$-Cauchy family, and $\mathcal{V} \in \beta$ be arbitrary. Since $\beta \subseteq \alpha$, $\mathcal{V} \in \alpha$, and $\mathcal{A}$ is an $\alpha$-Cauchy family, there are $V \in \mathcal{V}$, and $A \in \mathcal{A}$ such that $V \supseteq A$. Hence $\mathcal{A}$ is a $\beta$-Cauchy family so that $\bigcap \mathcal{A} \neq \emptyset$.

**Lemma E.** Let $X$ be an $E$-completely regular space and let $\mathcal{B}$ be a complemental base on $X$. Then the collection $\beta$ of all countable clopen coverings of $X$ from $\mathcal{B}$ is complete iff every $\mathcal{B}$-ultrafilter with the c.i.p. is fixed.

**Proof.** Necessity. Let $\mathcal{A}$ be an ultrafilter of $\mathcal{B}$ with the c.i.p. Suppose that $\mathcal{A}$ is not a $\beta$-Cauchy family. Then there would be a $\mathcal{V} \in \beta$ such that each $V_i \in \mathcal{V}$ does not meet some member of $\mathcal{A}$, namely, $A_i$. (For since
$\mathcal{A}$ is an ultrafilter, if $V_i$ meets each member of $\mathcal{A}$, then $V_i \in \mathcal{A}$. Thus, $\mathcal{A}$ would be a $\beta$-Cauchy family.) Hence $V_i \subseteq X \setminus A_i$ for each $i=1, 2, \cdots$. Then $X = \bigcup_{i=1}^{\infty} V_i = \bigcap_{i=1}^{\infty} \left( X \setminus A_i \right) = X \setminus \bigcap_{i=1}^{\infty} A_i$. This implies $\bigcap_{i=1}^{\infty} A_i = \varnothing$. This contradicts the fact that $\mathcal{A}$ has the c.i.p. Therefore, $\mathcal{A}$ is a $\beta$-Cauchy family, and $\bigcap \mathcal{A} \neq \varnothing$.

**Sufficiency.** Let $\mathcal{B}$ be a $\beta$-Cauchy family. Then there is a $\beta$-ultrafilter $\mathcal{A}$ containing $\mathcal{B}$. Since $\mathcal{B}$ is a $\beta$-Cauchy family and since $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{A}$ is a $\beta$-Cauchy family. Moreover, suppose that $A_1, A_2, \cdots$, are in $\mathcal{A}$ and have empty intersection. Then $\bigcup_{i=1}^{\infty} \left( X \setminus A_i \right) = X$, and $\mathcal{V} = \{ X \setminus A_i : i=1, 2, \cdots \} \subseteq \beta$ is in $\beta$. This would contradict the fact that $\mathcal{A}$ is a $\beta$-Cauchy family. Hence $\mathcal{A}$ is an ultrafilter with the c.i.p. and $\bigcap \mathcal{A} \neq \varnothing$. Therefore $\bigcap \mathcal{A} \neq \varnothing$.

**Theorem D.** Let $X$ be an $E$-completely regular space and let $\mathcal{F}$ be a complemental base on $X$ which is countably productive. Then the Wallman space $\eta(\mathcal{F})$ is $\mathcal{N}$-compact.

**Proof.** By Theorem A, $\mathcal{F}^{**}$ is a complemental base on $\eta(\mathcal{F})$. Lemma C says that every $\mathcal{F}^{**}$-ultrafilter with c.i.p. is fixed. Combining this with Lemmas D and E and Theorem C, $\eta(\mathcal{F})$ is $\mathcal{N}$-compact.

Note that an $E$-completely regular space is a Tychonoff space. Combining Theorem 3 of [4], and Theorem D, we have

**Corollary C.** Let $X$ be an $E$-completely regular space, let $\mathcal{F}$ be a complemental base on $X$ which is countably productive and suppose that $X$ is $\mathcal{F}$-realcompact. Then $X = \eta(\mathcal{F})$ and so $X$ is $\mathcal{N}$-compact.

**Remarks.** (1) Any discrete space has a complemental base which is countably productive.

(2) If $X = \mathcal{N}$, then the family $\mathcal{F}_1$ of all $E$-closed subsets which indeed is the family of all subsets is a complemental base which is countably productive. By [4, Theorem 3] $\eta(\mathcal{F}) = \nu X = \mathcal{N}$. However $\omega(\mathcal{F}) = \beta X$. Hence $\eta(\mathcal{F})$ is $\mathcal{N}$-compact but it is not $\mathcal{D}$-compact.

**References**


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