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EILENBERG-MAC LANE SPECTRA

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ABSTRACT. Let $K(Z_p)$ be the Eilenberg-Mac Lane spectrum with homotopy Z_p and let $A=H^*(K(Z_p); Z_p)$—the mod p Steenrod algebra. Let X be a locally finite spectrum. It is proven that

$$[K(Z_p), X] \to \mathrm{Hom}_A(H^*(X; Z_p), A)$$

is an isomorphism. It is also proven that there is a unique decomposition $X=(\oplus\ K(Z_p))\oplus Y$ where $H^*(Y; Z_p)$ as an A-module has no free summands.

0. **Introduction.** The primary purpose of this paper is to demonstrate the special role of Eilenberg-Mac Lane spectra in stable homotopy theory. Let $K(Z_p)$ denote the Eilenberg-Mac Lane spectrum associated to Z_p. It is the spectrum defining mod p cohomology denoted $H^*(\ ;Z_p)$ and its algebra of self-maps—the algebra of mod p stable cohomology operations—is the mod p Steenrod algebra A. It is trivially true that the homomorphism $[X, K(Z_p)]\to\mathrm{Hom}_A(H^*(K(Z_p); Z_p), H^*(X; Z_p))$, which assigns to each map its induced map in cohomology, is an isomorphism. But the algebra A possesses some rather special properties (see **[1]**, **[4]**, **[5]**) and because of this we can prove in addition:

THEOREM. *If X is bounded below and locally finite then the homomorphism*

$$[K(Z_p), X] \to \mathrm{Hom}_A(H^*(X; Z_p), H^*(K(Z_p); Z_p))$$

is an isomorphism.

We will also prove two parallel results, one algebraic and one topological, that further elucidate the special role of $K(Z_p)$.

THEOREM. *Let M be a bounded below A-module. Then M is isomorphic to $N\oplus F$ where F is a free A-module and N has no free summands, and, further, this decomposition is unique up to isomorphism.*

Let $X\oplus Y$ denote direct sum in the stable homotopy category.

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THEOREM. *Let X be a bounded below, locally finite spectrum. Then X is equivalent to $Y(\oplus KV)$ where V is a graded Z_p-vector space and $H^*(Y; Z_p)$ has no free summands, and further this decomposition is unique up to equivalence.*

1. **Modules over algebras like the Steenrod algebra.** Recall that a Poincaré algebra B over R is a connected algebra such that, for some n, there is an R-map $f: B_n \to R$ for which the bilinear pairing $B_r \times B_{n-r} \to R$ defined by $(a, b) \to f(ab)$ is nonsingular [5].

Let A be a connected algebra over a field which satisfies the following conditions:

(a) A is the union of a directed system of subalgebras $\{A_n\}$, $n \in I$, such that each A_n is a Poincaré algebra.

(b) A is flat as an A_n-module for each n.

EXAMPLES. (1) If A is a connected Hopf algebra which is the union of finite sub-Hopf algebras, then A is such an algebra (see [4] or [5]).

(2) In particular if A is the mod p Steenrod algebra then it is such an algebra.

(3) If A is a connected exterior algebra on generators $\{x_\alpha\}$, $\alpha \in \Lambda$, then A is such an algebra where I is the directed set of finite subsets of Λ and, for $n \in I$, A_n is the exterior algebra generated by x_α with $\alpha \in n$.

Let A be as above and let $\mathscr{M}_A$ be the category of bounded below A-modules (a Z-graded object M_* is bounded below if there is an m such that, for all $i \leq m$, $M_i = 0$).

THEOREM. *If F is a free A-module in $\mathscr{M}_A$ then F is injective in $\mathscr{M}_A$.*

This result was originally proven in the case in which A is the mod 2 Steenrod algebra by J. F. Adams and the author [1]. The general result is due to Moore and Peterson [5]—their version is somewhat more general than that stated here—and a different proof can be found in [4]. As an immediate corollary we have:

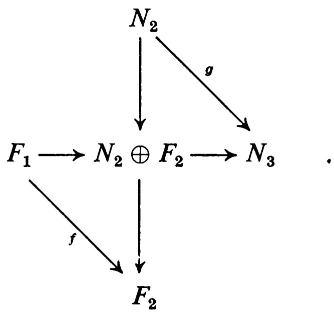
COROLLARY. *In $\mathscr{M}_A$ if F is free then* $\mathrm{Hom}_A(\ , F)$ *is exact.*

A further instance of the particularly simple role of free modules in $\mathscr{M}_A$ is the following:

THEOREM 1. *For any module M in $\mathscr{M}_A$, M is isomorphic to $N \oplus F$ where F is a free A-module and N has no free summands, and further this decomposition is unique up to isomorphism.*

PROOF. With no assumptions on the algebra A it is easily shown by the usual transfinite methods that there is an exact sequence $0 \to F \to M \to N \to 0$ where F is free and N has no free summands. But F is injective in $\mathscr{M}_A$ and therefore the sequence splits.

This gives the desired decomposition and we will now show that it is unique up to isomorphism. Suppose that we have an isomorphism $\alpha: N_1 \oplus F_1 \to N_2 \oplus F_2$ with F_1 and F_2 free, and N_1 and N_2 having no free summands. Let $j: F_1 \to N_2 \oplus F_2$ be the composite αi where i is the canonical inclusion. Then we have $0 \to F_1 \to N_2 \oplus F_2 \to^k N_3 \to 0$ exact and α induces an isomorphism of N_1 and N_3. Let $f: F_1 \to F_2$ be the composite Πj where Π is the projection and let $g: N_2 \to N_3$ be the composite kl where l is the canonical inclusion. We will prove that f is an isomorphism and the following commutative diagram then implies that g is an isomorphism:



We first prove that f is monic. Let F_1 have an A-base $\{x_i\}$ and F_2 an A-base $\{y_j\}$ and suppose that $f(\sum a_i x_i)=0$. If $j(x_i)=(m_i, z_i)$ then this implies that $j(\sum a_i x_i)=(\sum a_i m_i, 0)$. Our assumption that N_2 has no free summands implies that, for each m_i, there is a $b_i \neq 0$ in A such that $b_i m_i=0$. Since the summation is finite $a_i, b_i \in A_n$ for some n and all i. Since A_n is a Poincaré algebra there is an element c in A_n such that $ca_i m_i=0$ for all i and $ca_i \neq 0$ for some i. Then $c \sum a_i x_i \neq 0$ but $j(c \sum a_i x_i)=0$, which is a contradiction.

To show that f is epic it will suffice to show that y_j is in the image of f for all j. Suppose that $y_k \notin \operatorname{im} f$ for some y_k which we may assume to have minimal degree with this property. In any case $(0, y_k)=\alpha(n, x)$ which implies that $\alpha(n, 0)=(m, y')$ and $\alpha(0, x)=(m, y'')$ with either

$$y' = y_k + \sum_{j \neq k} a_j y_j \quad \text{and} \quad y'' = \sum_{j \neq k} a_j y_j,$$

or

$$y' = \sum_{j \neq k} a_j y_j \quad \text{and} \quad y'' = y_k + \sum_{j \neq k} a_j y_j.$$

In the first case since N_1 has no free summands there is an $a \neq 0$ in A such that $an=0$, in which case we have $0=\alpha(an, 0)=(am, ay') \neq 0$. In the second case either $\deg a_j > 0$ for all $j \neq k$ and, therefore, by our minimality assumption, $y_j \in \operatorname{im} f$ for $j \neq k$—but this implies that $y_k \in \operatorname{im} f$, or $\deg a_j = 0$ for some $j \neq k$, and we can argue as we did in the first case.

2. **Eilenberg-Mac Lane spectra.** Let $\mathscr{T}$ be the homotopy category of bounded below spectra constructed by Boardman [2], [3], [6] (in his terminology "highly connected" spectra). We will not need Boardman's construction of the objects and maps of $\mathscr{T}$ but we will use a number of the formal properties of this category, explicitly:

(a) $\mathscr{T}$ is additive and triangulated.

(b) If $f: X \to Y$ induces an isomorphism of homotopy or integral cohomology then f is an equivalence.

(c) If there is a common bound to the connectivity of X_α, $\alpha \in \Lambda$, then the direct sum $\bigoplus X_\alpha$ exists in $\mathscr{T}$.

(d) There is a functor $K: \mathscr{A}b \to \mathscr{T}$ which assigns to each abelian group G an Eilenberg-Mac Lane spectrum $K(G)$, this functor preserves sums, takes short exact sequences to exact triangles and induces an isomorphism $\mathrm{Hom}(G, H) \to [K(G), K(H)]^\circ$.

(e) For X in $\mathscr{T}$ there is a Postnikov tower $X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X$ and for Y in $\mathscr{T}$, $[Y, X] \to \mathrm{inj}\lim [Y, X_r]$ is onto.

A spectrum X is *locally finite* if $\Pi_r(X)$ is finitely generated for each r.

Fix a prime p and let $H^*(X) = [X, K(Z_p)]^*$, mod p cohomology, which we regard as a left module over $A = H^*(K(Z_p))$, the mod p Steenrod algebra. For a graded (bounded below) Z_p-vector space V, let $K(V) = \bigoplus_\alpha \Sigma^{d_\alpha} K(Z_p)$ where V has a basis $\{x_\alpha\}$ with $\deg x_\alpha = d_\alpha$.

The topological results of this paper are:

THEOREM 2. (a) *If X is locally finite and $\alpha: H^*(X) \to N \oplus F$ is an isomorphism of A-modules with F free over A then there are spectra Y and $K(V)$ such that $H^*(Y) = N$, $H^*(K(V)) = F$ and there is an equivalence $k: Y \oplus K(V) \to X$ such that $k^* = \alpha$.*

(b) *If X is locally finite then it is equivalent to $Y \oplus K(V)$ where $H^*(Y)$ has no free summands, and then Y and $K(V)$ are unique (up to equivalence).*

THEOREM 3. *If X is locally finite then*

$$[K(V), X] \to \mathrm{Hom}_A(H^*(X), H^*(K(V)))$$

is an isomorphism.

NOTE. (1) The restriction on X in Theorem 2(a) is essential. Let $X = K(Z_p) \oplus K(Z_p) \oplus \cdots$ then $H^*(X) = A \times A \times \cdots$ and $F = A \oplus A \oplus \cdots$ is a direct summand of $H^*(X)$ since free A-modules are injective in $\mathscr{M}_A$. But we observe that F is *not* realizable for, if $F = H^*(W)$, then

$$\mathrm{Hom}(H_0(W), Z_p) = H^\circ(W) = Z_p \oplus Z_p \oplus \cdots,$$

which cannot occur.

(2) I do not know if the restriction on X in Theorem 2(b) is essential.

(3) The restriction on X in Theorem 3 is essential. Let X be as in

Note (1), then the map $[K(Z_p), X]\to\mathrm{Hom}_A(H^*(X), A)$ is not an isomorphism. For we have that $[K(Z_p), X]^\circ=\mathrm{Hom}(Z_p, Z_p\oplus Z_p\oplus\cdots)$ which is countable but $\mathrm{Hom}^\circ_A(H^*(X), A)=\mathrm{Hom}^\circ_A(A\times A\times\cdots, A)$ which is uncountable. Note however that $[X, K(Z_p)]\to\mathrm{Hom}_A(A, H^*(X))$ is an isomorphism.

Theorems 2 and 3 will be proven by the following sequence of steps.

STEP 1. We prove Theorem 3 for spectra with finite Postnikov towers and from this prove that the map in Theorem 3 is epic in general.

STEP 2. We prove Theorem 2 using the results of Step 1.

STEP 3. We complete the proof of Theorem 3 using the results of Step 2.

STEP 1. Let $F=H^*(K(V))$, it is both projective and injective in $\mathcal{M}_A$. We begin by proving Theorem 3 for $X=K(G)$ where G is Z_p, Z_q for $q\neq p$ a prime or Z. The first case is immediate from the definitions of H^* and A. The second case follows from the fact that for p and q distinct primes $H^*(K(Z_q); Z_p)=0$. The third case is slightly more involved. From the short exact sequence $0\to Z\to Z\to Z_p\to 0$ we get the exact triangle $K(Z)\to K(Z)\to K(Z_p)\to K(Z)$ and from this we get the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(K(V); Z) & \longrightarrow & H^*(K(V)) & \longrightarrow & H^*(K(V); Z) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H_1} & & \downarrow{\scriptstyle H} & & \\ 0 & \to & \mathrm{Hom}_A(H^*(K(Z)), F) & \to & \mathrm{Hom}_A(A, F) & \to & \mathrm{Hom}_A(H^*(K(Z)), F) & \to & 0 \end{array}$$

The top row is an exact triangle but the map $H^*(K(V); Z)\to H^*(K(V); Z)$ is multiplication by p and $H^*(K(V); Z)$ is a Z_p-vector space i.e.

$$H_*(K(Z_p); Z) = H_*(K(Z); Z_p)$$

so applying the universal coefficient theorem we see that $H^*(K(Z_p); Z)$ is a Z_p-vector space, therefore so is $H^*(K(V); Z)=\Pi H^*(K(Z_p); Z)$. The bottom row is exact because $0\to H^*(K(Z))\to A\to H^*(K(Z))\to 0$ is the cohomology exact sequence of the triangle $K(Z)\to K(Z)\to K(Z_p)\to K(Z)$ and F is injective. We have already noted that H_1 is an isomorphism, it follows that H is an isomorphism.

Let X be a locally finite spectrum such that $\Pi_i(X)=0$ for $i>I$. Then $\Pi_*(X)$ is finitely generated and we will let size $\Pi_*(X)$ equal the dimension of $\Pi_*(X)\otimes Q$ plus the cardinality of Tor $\Pi_*(X)$. Then there is an exact triangle $\Sigma^k K(G)\leftarrow X_1\leftarrow X\leftarrow\Sigma^k K(G)$ with G either Z_q for some prime q or Z and size $\Pi_*(X_1)<$ size $\Pi_*(X)$ (unless $X=K(G)$ already). Then we have the following diagram:

$$\begin{array}{ccccccc} [K(V), K(G)] & \longleftarrow & [K(V), K_1] & \longleftarrow & [K(V), X] & \longleftarrow & [K(V), K(G)] \\ \downarrow{\scriptstyle H_1} & & \downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle H_3} & & \downarrow{\scriptstyle H_1} \\ \mathrm{Hom}_A(H^*(K(G)), F) & \longleftarrow & \mathrm{Hom}_A(H^*(X_1), F) & \longleftarrow & \mathrm{Hom}_A(H^*(X), F) & \longleftarrow & \mathrm{Hom}_A(H^*(K(G)), F) \end{array}$$

The top row is, in general, exact and, as above, F injective implies that the bottom row is exact. Then by induction on size we conclude that H_3 is an isomorphism.

Let X be a locally finite spectrum, then its Postnikov tower $X_0 \leftarrow X_1 \leftarrow \cdots X$ satisfies the condition that $\Pi_*(X_r)$ is finitely generated for each r. Therefore in the diagram

$$\begin{array}{ccc} [K(V), X] & \xrightarrow{L} & \lim [K(V), X_r] \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H_1} \\ \mathrm{Hom}_A(H^*(X), F) & \xrightarrow[L_1]{} & \lim \mathrm{Hom}_A(H^*(X_r), F) \end{array}$$

H_1 is an isomorphism. In general L is an epimorphism and since $H^*(X) = \mathrm{proj\,lim}\, H^*(X_r)$ the map L_1 is an isomorphism. Therefore H is an epimorphism for any locally finite spectrum.

PROOF OF THEOREM 2(a). Let $\alpha: H^*(X) \to N \oplus F$ be an isomorphism with F free over A. Since X is locally finite so is F and therefore $F = H^*(K(V))$ where $V = Z_p \otimes_A F$ and further

$$[X, K(V)] \to \mathrm{Hom}_A(H^*(K(V)), H^*(X))$$

is an isomorphism. Therefore there is an exact triangle $Y \to^g X \to^f K(V) \to Y$ with $f^* = \alpha^{-1} i_2$ and therefore $H^*(Y) = N$ and $g^* = \Pi_1 \alpha$. We have proven that $[K(V), X] \to \mathrm{Hom}_A(H^*(X), H^*(K(V)))$ is epic and therefore there is a map $h: K(V) \to X$ such that $h^* = \Pi_2 \alpha$. From this we get the commuting diagram:

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & X & \longrightarrow & K(V) & \longrightarrow & Y \\ \| & & \uparrow{\scriptstyle g+h} & & & & \| \\ Y & \longrightarrow & Y + K(V) & \longrightarrow & K(V) & \longrightarrow & Y \end{array}$$

and there exists a map $k: K(V) \to K(V)$ filling in the diagram. But $(g+h)^* = \alpha$ and therefore k^* is an isomorphism. Since $\Pi_*(K(V))$ is p-primary, k is an equivalence (if $K(V) \to^k K(V) \to W \to K(V)$ is exact then $\Pi_*(W)$ is p-primary and $H^*(W) = 0$), therefore $g+h$ is an equivalence.

PROOF OF THEOREM 2(b). By Theorem 1 there is an expression unique up to isomorphism $\alpha: H^*(X) \to N \oplus F$ with N having no free summands. By Theorem 2(a) X is equivalent to $Y \oplus K(V)$ with $H^*(Y) = N$ and $V = Z_p \otimes_A F$. Since $K(V)$ is determined up to equivalence by V it remains to show that Y is unique up to equivalence. So suppose we have equivalences $f: Y_1 \oplus K(V) \to X$ and $g: Y_2 \oplus K(V) \to X$ with $\alpha = f^* = g^*$. Then we get

$$\begin{array}{ccccccc} K(V) & \leftarrow & Y_1 \oplus K(V) & \leftarrow & Y_1 & \leftarrow & K(V) \\ \| & & \uparrow{\scriptstyle f^{-1}g} & & & & \| \\ K(V) & \leftarrow & Y_2 \oplus K(V) & \leftarrow & Y_2 & \leftarrow & K(V) \end{array}$$

which commutes since $[W, K(V)] \to \mathrm{Hom}_A(H^*(K(V)), H^*(W))$ is an isomorphism. Therefore a fill-in $h\colon Y_2 \to Y_1$ exists and is an equivalence.

PROOF OF THEOREM 3. It remains to show that

$$[K(V), X] \to \mathrm{Hom}_A(H^*(X), F)$$

is a monomorphism. Let us suppose that we are given $f\colon K(V) \to X$ with $f^*=0$ (in Z_p-cohomology). We have an exact triangle $K(V) \to^f X \to Y \to K(V)$ which in Z_p-cohomology gives the short exact sequence $0 \to F \to H^*(Y) \to H^*(X) \to 0$. Therefore $H^*(Y)=H^*(X)\oplus F$ and by Theorem 2(a) there is an equivalence $g\colon Y \to X \oplus K(V)$. This gives commutative diagram

$$\begin{array}{ccccccc} K(V) & \longleftarrow & Y & \longleftarrow & X & \xleftarrow{\;f\;} & K(V) \\ \| & & \downarrow{\scriptstyle g} & & & & \| \\ K(V) & \longleftarrow & X \oplus K(V) & \longleftarrow & X & \xleftarrow[0]{} & K(V) \end{array}$$

and so there is a fill-in $h\colon X \to X$. But g an equivalence implies that h is an equivalence and therefore $f=0$.

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