SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

0-DIVISORS IN GROUP RINGS

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Abstract. If \( G \) is any group with two finite subgroups \( H, K, K \leq G, (|H|, |K|)=1 \), then \( RG \) has no \( 0 \)-divisors congruent to \( 1 \) modulo the augmentation ideal.

Let \( R \) be a commutative unitary ring of characteristic 0, and \( \mathcal{G} \) denote the augmentation ideal \( \mathcal{A}(G) \) of the group ring \( RG \) for a group \( G \). If \( G \) is finite and not of prime-power order then, as J. Roseblade and R. Phillips have recently proved (unpublished), \( RG \) contains a \( 0 \)-divisor congruent to 1 modulo \( \mathcal{G} \). Their proof depends heavily on properties of Schmidt groups. We give here a simple proof generalizing this result to the infinite case. For \( T \leq G \), let \( \mathcal{A}(T) \) be the left ideal in \( RT \), generated by \( \{t-1|t \in T\} \).

Theorem. Let \( G \) be a group containing two finite subgroups \( H \) and \( K \) where \( H \leq N_G(K) \) and \( (|H|, |K|)=1 \). Then \( \mathcal{A}(K) \cdot \mathcal{A}(H) \cdot x = 0 \) for some \( x \equiv 1 \mod \mathcal{G} \) in \( RG \).

Proof. Let \( y = \sum_{h \in H} h, z = \sum_{k \in K} k, |H| = m, |K| = n, \) where \( (m, n)=1 \). Then \( k \in K \Rightarrow (k-1)z = 0 \), and similarly \( h \in H \Rightarrow (h-1)y = 0 \). Since \( (m, n)=1 \), there exist \( r, s \) in \( \mathbb{Z} \) (and hence in \( R \)) such that \( rm + sn = 1 \). Put \( x = ry + sz \). If \( \rho : RG \rightarrow R \) is the augmentation map, then \( \rho(x) = r \rho(y) + s \rho(z) = rm + sn = 1 \), so that \( x \equiv 1 \mod \mathcal{G} \), since \( \mathcal{G} = \text{kernel } \rho \). Also

\[
\mathcal{A}(K) \cdot \mathcal{A}(H) \cdot x = r \cdot \mathcal{A}(K) \cdot \mathcal{A}(H)y + s \cdot \mathcal{A}(K) \cdot \mathcal{A}(H)z
\]

\[
= 0 + s \cdot \mathcal{A}(K) \cdot \mathcal{A}(H) \cdot z, \quad \text{since } \mathcal{A}(H)y = 0,
\]

\[
= s \cdot \mathcal{A}(K) \cdot z \cdot \mathcal{A}(H), \quad \text{since } H \leq N_G(K),
\]

\[
= 0, \quad \text{since } \mathcal{A}(K) \cdot z = 0.
\]

This proves the result. Q.E.D.

Since the existence of such \( 0 \)-divisors easily implies that the intersection of all powers of the augmentation ideal is not 0, we have:

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**Corollary.** If $G$ has a finite subgroup which is not of prime-power order then $\bigcap_\alpha G^\alpha \neq 0$.

**Proof.** We may suppose that $G$ is finite and not of prime-power order. It suffices to show that $G$ has subgroups $H, K \neq 1$ of relatively prime order with $H \leq N_G(K)$.

Let $P$ be a $p$-Sylow subgroup of $G$. If $G$ has a normal $p$-complement $K$, take $P = H$. If not, there exists a subgroup $K \neq 1$ in $P$, such that $N_G(P)/C_G(P)$ is not a $p$-group. Take $H$ to be a $q$-subgroup of $N_G(K)$ for some $q \neq p$.

With these subgroups $H$ and $K$, we can now apply the Theorem and the comment above to complete the proof of the Corollary. Q.E.D.

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