INVARIANT SUBSPACES FOR PRODUCTS OF HERMITIAN OPERATORS

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ABSTRACT. It is shown that a nonscalar operator which is the product of a Hermitian operator and a positive operator has a nontrivial hyperinvariant subspace; this is a slight generalization of a result of Suzuki's.

Suzuki [3] has shown that an operator $T$ which is the product of two positive operators has a nontrivial hyperinvariant subspace; (i.e., a closed linear manifold, different from $\{0\}$ and the entire space, which is left invariant by every operator which commutes with $T$). We present a simple proof of a slight generalization of Suzuki's result.

THEOREM. If $T$ is the product of a positive operator and a Hermitian operator, (and $T$ is not a multiple of the identity), then $T$ has a nontrivial hyperinvariant subspace.

PROOF. Assume that $T = RK$ where $R$ is positive and $K$ is Hermitian; (if the product occurs in the other order simply apply the following to $T^*$). If $R$ or $K$ has a nontrivial nullspace then so does $T$ or $T^*$, and the nullspace of an operator is hyperinvariant. Assume that $R$ and $K$ are injective; they then also have dense ranges since they are Hermitian. Now $R$ has a unique positive square root $R^{1/2}$. It is trivial to verify the fact that

$$TR^{1/2} = R^{1/2}(R^{1/2}KR^{1/2})$$

and

$$(R^{1/2}K)T = (R^{1/2}KR^{1/2})(R^{1/2}K).$$

Since $R^{1/2}K$ and $R^{1/2}$ are injective and have dense ranges, the operator $T$ is quasi-similar to the Hermitian operator $R^{1/2}KR^{1/2}$ and has a nontrivial hyperinvariant subspace by the well-known result of Sz. Nagy-Foiaş (cf. [2], [1, p. 103]).

It appears to be much more difficult to prove that the product of two Hermitian operators has an invariant subspace; if the above were extended

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to the product of a positive operator and a unitary operator then, by the
polar decomposition, the invariant subspace problem would be solved.

REFERENCES


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