CONVOLUTIONS OF CONTINUOUS MEASURES
AND SUMS OF AN INDEPENDENT SET

JAMES MICHAEL RAGO

Abstract. Let $E$ be a compact independent subset of an l.c.a. group $G$; $\mu_1, \ldots, \mu_{n+1}$ continuous regular bounded Borel measures on $G$; and $k_1, \ldots, k_n$ integers. Let $k_i \times E = \{k_i x | x \in E\}$. We prove

(1) $\mu_1 \ast \cdots \ast \mu_{n+1}(k_1 \times E + \cdots + k_n \times E) = 0$ (the proof is a combinatorial argument).

As a corollary of (1) we obtain (2) if $H$ is any closed nondiscrete subgroup of $G$, then the intersection of $H$ with the group generated by $E$ has zero $H$-Haar measure.

1. In [2, Theorem 2], Hartman and Ryll-Nardzewski showed neatly that if $\mu_1$ and $\mu_2$ are continuous bounded measures on $T$, the circle group, and $E \subseteq T$ is a compact independent subset of $T$, then $\mu_1 \ast \mu_2(E) = 0$. In a more complicated manner, Salinger and Varopoulos proved this result for any metrizable group $G$ in [4, Theorem 1]. We prove a generalization of this theorem which holds for an arbitrary nondiscrete l.c.a. group $G$.

We denote by $Z$ the set of all integers, $Z^+ = \{k \in Z | k > 0\}$. A set $E \subseteq G$ is independent if the equation $k_1 e_1 + \cdots + k_n e_n = 0$, $k_i \in Z$, $e_i \in E$, yields $k_i e_i = 0$ for all $i = 1, \ldots, n$ or the $e_i$'s are not distinct. Note that an independent set may contain $0$. We set $k \times E = \{k e | e \in E\}$, and $nE = \{e_1 + \cdots + e_n | e_i \in E\}$ for $k \in Z$, $n \in Z^+$; $E_1 + E_2 = \{e_1 + e_2 | e_1 \in E_1, e_2 \in E_2\}$.

Here is our main result.

Theorem. If $G$ is a locally compact abelian group, $E \subseteq G$ a compact independent subset, $\mu_1, \ldots, \mu_{n+1} \in M(G)$ positive continuous measures, $k_i \in Z$, $i = 1, \ldots, n$, and $x_0 \in G$, then

$$\mu_1 \ast \cdots \ast \mu_{n+1}(k_1 \times E + \cdots + k_n \times E - x_0) = 0.$$ 

At the end of the proof of the theorem, we will derive the following corollary:

Corollary. If $G$ is an l.c.a. group, $H \subseteq G$ is a closed nondiscrete subgroup with Haar measure $h$, $x_0 \in G$, and $E \subseteq G$ is compact independent, then $h((GpE - x_0) \cap H) = 0$, where $GpE = \bigcup_{n=1}^{\infty} n(E \cup -E)$.

Received by the editors June 12, 1973.


Key words and phrases. Compact independent set, Haar measure.

1 This work formed a part of the author's Ph.D. dissertation, which was written at Northwestern University under the supervision of Colin C. Graham.
2. Proof of Theorem. The proof is by induction. If \( n=1 \), we follow the proof of [2, Theorem 2]. Suppose \( k_1=1 \). By definition,

\[
\mu_1 * \mu_2(E) = \int_G \mu_1(E-x) \, d\mu_2(x).
\]

For \( x_1 \neq x_2 \), suppose there is a fixed element \( y_0 \in E-x_1 \cap E-x_2 \); then \( y_0 = e_0 - x_1 = f_0 - x_2 \), where \( e_0, f_0 \in E \) are fixed. Hence,

\[
\begin{align*}
(2.1) & \quad x_1 + y_0 = e_0; \\
(2.2) & \quad x_2 + y_0 = f_0.
\end{align*}
\]

If \( y \in E-x_1 \cap E-x_2 \) is arbitrary, then \( y = e-x_1 = f-x_2 \) for certain \( e, f \in E \), so

\[
\begin{align*}
(2.3) & \quad x_1 + y = e; \\
(2.4) & \quad x_2 + y = f.
\end{align*}
\]

The equation (symbolically written) \((2.1)-(2.2) = (2.3)-(2.4)\) yields \( e_0 - f_0 = e - f \). Since either \( e_0 \) or \( f_0 \) is nonzero (otherwise \( x_1 = x_2 \)), the independence of \( E \) requires one of the following four possibilities: \( e_0 = e, e_0 = f, f_0 = e, \) or \( f_0 = f \). By (2.3) and (2.4), these imply

\[
y = e_0 - x_1, \quad y = f_0 - x_1, \quad y = e_0 - x_2, \quad \text{or} \quad y = f_0 - x_2
\]

respectively. Since \( e_0, f_0, x_1, \) and \( x_2 \) are fixed, this shows

\[
\text{card}[E-x_1 \cap E-x_2] \leq 4
\]

and \( \mu_1(E-x_1 \cap E-x_2) = 0 \), by continuity of \( \mu_1 \). (Since \( e_0 - x_1 = f_0 - x_2 = y_0 \), we actually have \( \text{card} \leq 3 \).) The set of \( x \) with \( \mu_1(E-x) > 0 \) is then countable, since it is easy to see that otherwise \( \|\mu_1\| = \infty \). Hence by (1) and the continuity of \( \mu_2 \), we have \( \mu_1 * \mu_2(E) = 0 \). If \( x_0 \in G \), \( \mu_1 * \mu_2(E-x_0) = [(\delta_{x_0} * \mu_1) * \mu_2] = 0 \), where \( \delta_{x_0} \) is the point mass at \( x_0 \). Finally, in the case \( k_1 \neq 1 \), \( k_1 \times E \) is also compact independent, so the Theorem holds for \( n=1 \).

We now suppose the Theorem holds for \( n-1 \) and all \( k_1, \ldots, k_{n-1} \in Z \); we show it holds for \( n \). Let \( \mu_1, \ldots, \mu_{n+1}, k_1, \ldots, k_n \) and \( x_0 \) be given. Suppose

\[
\mu_1 * \cdots * \mu_{n+1}(k_1 \times E + \cdots + k_n \times E) = \delta > 0.
\]

**Lemma.** \( E \) may be written as the finite disjoint union of Borel sets \( E_i, \)

\[
i=1, \ldots, p, \text{ such that for some choice of integers } i_1, \ldots, i_n, 0 \leq i_1 < \cdots < i_n \leq p, \text{ we have}
\]

\[
\mu_1 * \cdots * \mu_{n+1}(k_1 \times E_{i_1} + \cdots + k_n \times E_{i_n}) > 0.
\]
PROOF OF LEMMA. We first set $k = \sum_{1}^{n} |k_{i}|$. By the inductive hypothesis of the Theorem, for each choice of integers $l_{1}, \ldots, l_{n-1}$, we have

$$\mu_{1} \ast \cdots \ast (\mu_{n} \ast \mu_{n+1})\left(\sum_{1}^{n-1} l_{i} \times E\right) = 0.$$  

Containing each set $\sum_{1}^{n-1} l_{i} \times E$ there is, therefore, by the regularity of $\mu_{1} \ast \cdots \ast \mu_{n+1}$, an open set $U_{1}, \ldots, l_{n-1}$ with $\sum_{1}^{n-1} l_{i} \times E \subseteq U_{1}, \ldots, l_{n-1}$, and

$$(5) \quad \mu_{1} \ast \cdots \ast \mu_{n+1}(U_{1}, \ldots, l_{n-1}) < \delta|\mathcal{A}|,$$

where $\mathcal{A}$ is the number of possible choices of $l_{1}, \ldots, l_{n-1}$ with $\sum_{1}^{n-1} |l_{i}| \leq k$. (For example, $\mathcal{A} \leq (2k + 1)^{n-1}$.)

For each $U_{1}, \ldots, l_{n-1}$, there is a relatively compact neighborhood $V$ of 0 such that

$$(6) \quad \sum_{1}^{n-1} l_{i} \times E + kV \subseteq U_{1}, \ldots, l_{n-1},$$

by compactness. Taking (finite) intersections when necessary, we may assume (6) holds with one $V$ for all choices of $(l_{1}, \ldots, l_{n-1})$ with $\sum_{1}^{n-1} |l_{i}| \leq k$.

Since $E$ is compact, it is covered by finitely many (say $p$) neighborhoods $(e_{i} + V) \cap E$, with $e_{i} \in E$, $i=1, \ldots, p$ ($p$ depending on $V$). Define $E_{i} = (e_{i} + V) \cap E$

$$E_{i} = (e_{i} + V) \cap E \setminus ((E_{1} \cup \cdots \cup E_{i-1}), \quad i = 2, \ldots, p,$$

so the $E_{i}$'s are disjoint Borel sets with union $E$. Each set of the form

$$(7) \quad F = k_{1} \times E_{i_{1}} + \cdots + k_{n} \times E_{i_{n}}, \quad 1 \leq i_{1}, \cdots, i_{n} \leq p,$$

is Borel. We claim that at least one of the sets of form (7), with $i_{1} < \cdots < i_{n}$ all distinct, has strictly positive $\mu_{1} \ast \cdots \ast \mu_{n+1}$-measure. This will prove the lemma.

Indeed, consider any set of form (7) with at least two subscripts equal, say $i_{1} = i_{2}$. It may then be written

$$F = k_{1} \times E_{i_{1}} + k_{2} \times E_{i_{1}} + k_{3} \times E_{i_{3}} + \cdots + k_{n} \times E_{i_{n}}$$

$$\subseteq (e_{i_{1}} + V) + k_{2} \times (e_{i_{3}} + V) + k_{3} \times (e_{i_{4}} + V)$$

$$+ \cdots + k_{n} \times (e_{i_{n}} + V)$$

$$\subseteq (k_{1} + k_{2}) \times E + \cdots + k_{n} \times E + kV$$

$$\subseteq U_{(k_{1} + k_{2}, k_{3}, \ldots, k_{n})},$$

by (6). Hence the union of such sets (with two subscripts equal) is contained in the union of the $(A)$ sets $U_{l_{1}, \ldots, l_{n-1}}$, and thus has $\mu_{1} \ast \cdots \ast \mu_{n+1}$-measure less than $\delta$, by (5). This completes the proof of the Lemma.
For convenience we will assume, relabelling if necessary, that \( i_1 = 1, \ldots, i_n = n \). We write

\[
\mu_1 \ast \cdots \ast \mu_{n+1}\left(\sum_{i=1}^{n} k_i \times E_i\right)
\]

(8)

\[
= \int \mu_1 \ast \cdots \ast \mu_n\left(\sum_{i=1}^{n} k_i \times E_i - x\right) \, d\mu_{n+1}(x).
\]

We claim that

\[
E_{x_1} \neq E_{x_2} \Rightarrow \mu_1 \ast \cdots \ast \mu_n\left(\sum_{i=1}^{n} k_i \times E_i - x_1 \cap \sum_{i=1}^{n} k_i \times E_i - x_2\right) = 0.
\]

Then as in the proof of the case \( n = 1 \), we have \( \mu_1 \ast \cdots \ast \mu_n\left(\sum_{i=1}^{n} k_i \times E_i - x\right) = 0 \) except for perhaps countably many \( x \), and we obtain, by (8),

\[
\mu_1 \ast \cdots \ast \mu_{n+1}\left(\sum_{i=1}^{n} k_i \times E_i\right) = 0,
\]

contradicting (4) and hence (3). We will thus obtain

\[
\mu_1 \ast \cdots \ast \mu_{n+1}\left(\sum_{i=1}^{n} k_i \times E\right) = 0.
\]

To prove the claim, suppose there is a fixed \( y_0 \in (\sum_{i=1}^{n} k_i \times E_i - x_1) \cap (\sum_{i=1}^{n} k_i \times E_i - x_2) \). Then

\[
x_1 + y_0 = k_1 e_1^0 + \cdots + k_n e_n^0,
\]

(11.1)

\[
x_2 + y_0 = k_1 f_1^0 + \cdots + k_n f_n^0,
\]

(11.2)

\( e_i^0, f_i^0 \in E_i \) fixed. Given an arbitrary \( y \in (\sum_{i=1}^{n} k_i \times E_i - x_1) \cap (\sum_{i=1}^{n} k_i \times E_i - x_2) \), we have

\[
x_1 + y = k_1 e_1 + \cdots + k_n e_n,
\]

(11.3)

\[
x_2 + y = k_1 f_1 + \cdots + k_n f_n,
\]

(11.4)

\( e_i, f_i \in E_i \). For at least one value of \( i \), say \( i = j \), we must have \( k_j e_j \neq k_j f_j \), since otherwise \( x_1 = x_2 \). Hence the equation denoted by (11.1)–(11.2)= (11.3)–(11.4) and the disjointness of the \( E_i \) 's with the independence of \( E \) yields

\[
k_j e_j^0 - k_j f_j^0 - k_j e_j + k_j f_j = 0.
\]

Since one of the first two terms of (12) must be nonzero, we must have (again by independence) that one of \( e_j \) or \( f_j \) must equal \( e_i^0 \) or \( f_i^0 \), the latter two being fixed. Then by (11.3) and (11.4), either \( y = k_j e_j^0 + \sum_{i \neq j} k_i e_i - x_1 \), \( y = k_j f_j^0 + \sum_{i \neq j} k_i f_i - x_1 \), \( y = k_j e_j^0 + \sum_{i \neq j} k_i e_i - x_2 \), or \( y = k_j f_j^0 + \sum_{i \neq j} k_i f_i - x_2 \);
hence we have

\[ y \in \bigcup_{s=1,2} \left( k_j e_j^0 + \sum_{i \neq j} k_i \times E_i - x_s \right) \cup \bigcup_{s=1,2} \left( k_j f_j^0 + \sum_{i \neq j} k_i \times E_i - x_s \right). \]

Since \( y \) is arbitrary, \( (\sum k_i \times E_i - x_1) \cap (\sum k_i \times E_i - x_2) \) is contained in a finite union (over \( j = 1, \cdots, n \)) of sets of the above form; however, by the inductive hypothesis,

\[ \mu_1 \ast \cdots \ast \mu_{n+1} \left( k_j e_j^0 + \sum_{i \neq j} k_i \times E_i - x_s \right) = \left( \mu_1 \ast \mu_2 \right) \ast \cdots \ast \mu_{n+1} \left( \sum_{i \neq j} k_i \times E_i - (x_s - k_j e_j^0) \right) = 0, \]

(there are \( n \)-convolutions and \( n-1 \)-summands); \( s = 1, 2 \), and similarly for \( f_j^0 \). Hence the claim, and (10), are proved. (We note it is not generally possible to have \( \text{card} \{ \sum k_i \times E_i - x_1 \cap \sum k_i \times E_i - x_2 \} < \infty \); for example, choose \( E \) infinite and fix \( x_1, x_2 \in E \); then \( E \subseteq E + E - x_1 \cap E + E - x_2 \).)

Finally,

\[ \mu_1 \ast \cdots \ast \mu_{n+1} \left( \sum_{i=1}^n k_i \times E - x_0 \right) = (\delta_{x_0} \ast \mu_1) \ast \cdots \ast \mu_{n+1} \left( \sum_{i=1}^n k_i \times E - x_0 \right) = 0, \]

and the Theorem is proved. \( \text{Q.E.D.} \)

Note that the requirement that \( \mu_i \) is positive may be lifted by passing to the total variation measure.

If \( G \) is a compact nondiscrete group with Haar measure \( h \), \( E \subseteq G \) compact independent, and \( Gp(E) = \bigcup_{n=1}^{\infty} n(E \cup -E) \), then our Theorem implies at once that \( h(GpE) = 0 \). Indeed, \( h \) is a bounded, continuous, idempotent measure, and thus annihilates \( \sum_{i=1}^n k_i \times E \) for all \( n \) and \( k_i \). Our Corollary is a generalization of this, due to Graham [1] (who strengthened Rudin [3, 5.3.6]), giving a proof considerably simpler than Graham’s.

3. Proof of Corollary. Given a closed set \( S \subseteq G \), define \( h|_S(A) = h(A \cap S \cap H) \) for all \( A \subseteq G \), \( A \) Borel. In particular, we may write \( h|_H = h \), and thus consider \( h \) as a (possibly unbounded) measure on \( G \). We claim that if \( S \subseteq H \) and \( A \subseteq G \) are compact sets, then setting \( B = A - nS \) for any \( n \in \mathbb{Z}^+ \), we obtain

\[ h|_B \ast h|_S \ast \cdots \ast h|_S(A) = h(S)^n h(A), \]
where there are $n$-convolutions of $h|_S$. Indeed,

$$h|_B * h|_S * \cdots * h|_S(A)$$

$$= \int h|_B(A - x_1 - \cdots - x_n) \, dh|_S(x_1) \cdots dh|_S(x_n).$$

Since $A - x_1 - \cdots - x_n \subset A - nS = B$ for $x_i \in S$, $i=1, \cdots, n$,

$$h|_B(A - x_1 - \cdots - x_n) = h(A - x_1 - \cdots - x_n) = h(A)$$

because $x_i \in S \subset H$ implies $(A - x_1 - \cdots - x_n) \cap H = A \cap H - x_1 - \cdots - x_n$. Thus,

$$\int h|_B(A - x_1 - \cdots - x_n) \, dh|_S(x_1) \cdots dh|_S(x_n)$$

$$= \int h(A) \, dh|_S(x_1) \cdots dh|_S(x_n)$$

$$= h(S)^n h(A),$$

and the claim is proved.

Suppose now that $E$ and $k_1, \cdots, k_n$ are given; we fix $W \subset H$ to be any relatively compact neighborhood in $H$, and set $A = k_1 \times E + \cdots + k_n \times E - x_0$, $S = \overline{W}$, $B = A - nS$ as above. Since $h|_B$ and $h|_S$ are then bounded continuous measures on $G$, our Theorem implies

$$h|_B * h|_S * \cdots * h|_S(A) = 0,$$

where there are $n$-convolutions of $h|_S$. However, (13) implies that $h(S)^n h(A) = 0$. Since $\overline{W} = S$, then $h(S) \neq 0$ and it follows that $h(A) = 0$. Thus $h(((GpE - x_0) \cap H) = 0$. Q.E.D.

I would like to express my gratitude to the referee for his helpful remarks and criticisms.

References


Department of Mathematics, Northwestern University, Evanston, Illinois 60201

Current address: 6507 North Kedzie Avenue, Chicago, Illinois 60645