AUTOMORPHISMS AND TENSOR PRODUCTS OF ALGEBRAS

JOHN W. BUNCE

Abstract. In this note we prove that if $A$ is a complex Banach algebra with identity, then the automorphism on $A \hat{\otimes} A$ determined by $\theta(a \otimes b) = b \otimes a$ is inner if and only if $A = M_n(C)$.

Let $A$ be a $C^*$-algebra and let $A \otimes^* A$ be the $C^*$-tensor product of $A$ with itself. Let $\theta$ be the automorphism of $A \otimes^* A$ determined by $\theta(a \otimes b) = b \otimes a$ for all $a, b$ in $A$. S. Sakai proved in [3] that $\theta$ is inner if and only if $A$ is the algebra $M_n(C)$ of all $n \times n$ complex matrices for some positive integer $n$. In an invited talk at the International Conference on Banach Spaces, Wabash College, June, 1973, Sakai asked if the theorem had an extension to general Banach algebras. In Theorem 1 of this note we prove that if $A$ is a complex Banach algebra with identity of norm one, then the automorphism on $A \hat{\otimes} A$ determined by $\theta(a \otimes b) = b \otimes a$ is inner if and only if $A = M_n(C)$. In Theorem 2 we prove that if $A$ is an algebra over an algebraically closed field $F$, then the automorphism $\theta$ as defined above on the algebraic tensor product $A \otimes_F A$ is inner if and only if $A = M_n(F)$. The proofs of these theorems are much easier than the proof of the $C^*$-algebra theorem.

Let $A$ be a Banach algebra with identity $e$; we assume $\|e\| = 1$. Let $A \hat{\otimes} A$ be the algebraic tensor product of the complex vector space $A$ with itself, and let $A \hat{\otimes} A$ be the completion of $A \otimes A$ in the greatest crossnorm [4]. The greatest crossnorm on $A \otimes A$ is an algebra norm [1], so $A \hat{\otimes} A$ is a Banach algebra. It is clear that the map $\theta$ from $A \hat{\otimes} A$ to $A \hat{\otimes} A$ determined by $\theta(a \otimes b) = b \otimes a$, for all $a, b$ in $A$, is an automorphism. If $A = M_n(C)$, then $A \hat{\otimes} A$ is algebraically isomorphic to $M_n(C)$, and it is well known that every automorphism of $M_n(C)$ is inner.

Theorem 1. If the automorphism $\theta$ of $A \hat{\otimes} A$ is inner, then $A = M_n(C)$ for some positive integer $n$.

Received by the editors August 9, 1973.

AMS (MOS) subject classifications (1970). Primary 46H20, 46M05; Secondary 16A72, 16A40.

Key words and phrases. Banach algebra, tensor product, automorphism.

The author was partially supported by NSF Grant GP-37526.
Proof. We first prove that $A$ is a simple algebra. Let $I$ be a closed ideal in $A$ with $I \neq A$. Let $p$ be the quotient map from $A$ to $A/I$ and consider the canonical map $\widehat{\otimes} \text{id}: A \otimes A \rightarrow (A/I) \otimes A$, where $\text{id}$ is the identity map of $A$. Then $p \otimes \text{id}$ is clearly a homomorphism, and by [5, p. 445] $\text{Ker}(p \otimes \text{id}) = \text{cl}(I \otimes A)$, the closure of the space spanned by the elementary tensors $b \otimes a$, $b \in I$, $a \in A$. Since $\theta$ is inner we have that $\theta(\text{cl}(I \otimes A)) \subseteq \text{cl}(I \otimes A)$. If $a \in I$ we then have $\theta(a \otimes e) \in \text{Ker}(p \otimes \text{id})$, so $(p \otimes \text{id})(e \otimes a) = (e+I) \otimes a = 0$, and $a = 0$. Hence $I = \{0\}$ and $A$ has no nontrivial closed ideals, and in fact no nontrivial ideals since $A$ is complete and has an identity. By the classical Wedderburn-Artin theorem [2, Theorem 2.1.8], we now need only show that $A$ is finite dimensional. Let $d \in A \otimes A$ implement the inner automorphism $\theta$, so that $b \otimes a = d(a \otimes b)d^{-1}$ for all $a, b \in A$. Now choose $z$ and $w$ in the algebraic tensor product $A \otimes A$ with the property that

\[
\|z - d\| < (4 \|d^{-1}\|)^{-1}, \quad \|z\| < \|d\| + 1,
\]

\[
\|w - d^{-1}\| < (4(\|d\| + 1))^{-1}.
\]

Let $z = \sum_{i=1}^{r} x_i \otimes y_i$, $w = \sum_{i=1}^{s} u_i \otimes v_i$. Then if $a, b$ are in $A$ we have

\[
\|b \otimes a - z(a \otimes b)w\| \leq \|b \otimes a - d(a \otimes b)d^{-1}\|
\]

\[
+ \|d(a \otimes b)d^{-1} - z(a \otimes b)d^{-1}\|
\]

\[
+ \|z(a \otimes b)d^{-1} - z(a \otimes b)w\|
\]

which is less than or equal to $(\|a\| \|b\|)/2$. Thus for all $a$ in $A$ we have

\[
\|e \otimes a - z(a \otimes e)w\| \leq \|a\|/2,
\]

and hence for all $a$ in $A$,

(*) \quad \left\| e \otimes a - \sum x_i a u_j \otimes y_i v_j \right\| \leq \|a\|/2,

where the sum is over all $1 \leq i \leq r$, $1 \leq j \leq s$.

Now if $f \in A^*$, the bilinear function from $A \times A$ to $A$ defined by $(a, b) \mapsto f(a)b$ determines a linear function $F$ from $A \otimes A$ to $A$ with the property that $\|F\| = \|f\|$ and $F(a \otimes b) = f(a)b$ for all $a, b$ in $A$ [5, Proposition 43.12]. Choose $f \in A^*$ such that $f(e) = 1 = \|f\|$ and apply the corresponding $F$ to the equation (*) to obtain

\[
\left\| a - \sum f(x_i a u_j) y_i v_j \right\| \leq \|a\|/2,
\]

for all $a \in A$. Now let $H$ be the finite-dimensional space spanned by the set \{ $y_i v_j : 1 \leq i \leq r$, $1 \leq j \leq s$ \}. We have shown that for all $a \in A$, $H \cap B(a, \|a\|/2) \neq \emptyset$, where $B(a, \|a\|/2)$ is the closed ball of radius $\|a\|/2$ and center $a$. But by Riesz's lemma [6, p. 84], this fact forces $H$ to equal $A$. Thus $A$ is finite dimensional. Q.E.D.
Theorem 2. If $A$ is an algebra with identity $e$ over an algebraically closed field $F$ and the automorphism of the algebraic tensor product $A \otimes_F A$ determined by $\theta(a \otimes b) = b \otimes a$ is inner, then $A = M_n(F)$ for some positive integer $n$.

Proof. The proof that $A$ is a simple algebra is almost the same as the proof that $A$ is simple in Theorem 1; we omit the details. Let $d \in A \otimes A$ be such that $b \otimes a = d(a \otimes b)d^{-1}$ for all $a, b \in A$. Let $d = \sum_{i=1}^{n} x_i \otimes y_i$, where we assume that the set $\{x_i\}$ is linearly independent [4, Lemma 1.1]. We will show that $A$ is the linear span of $\{y_j; 1 \leq j \leq n\}$. For $a, b$ in $A$ we have

$$\left(\sum x_i \otimes y_i\right) a \otimes b = b \otimes a \left(\sum x_i \otimes y_i\right).$$

Now if for some $a \in A$ and index $i$, $ay_i \notin \text{span}\{y_j\}$, choose $g$ in the algebraic dual $A'$ of $A$ such that $g(ay_i) = 1$, $g(y_j) = 0$ for all $j$. Let $G: A \otimes A \to A$ be the linear function determined by the bilinear function $(c, b) \to g(b)c$ on $A \times A$, set $b = e$ in (**), and apply $G$ to obtain

$$\sum g(y_j)x_ja = \sum g(ay_j)x_j.$$

Hence $\sum g(ay_j)x_j = 0$ but $g(ay_i) \neq 0$, which contradicts our assumption that the $\{x_i\}$ were linearly independent. Thus $\text{span}\{y_j\}$ is a left ideal, and a symmetrical argument shows that $\text{span}\{y_j\}$ is a two-sided ideal. But $A$ is simple, so $A = \text{span}\{y_j\}$ and is finite dimensional. The Wedderburn-Artin theorem again implies that $A = M_n(F)$. Q.E.D.

References


Department of Mathematics, University of Kansas, Lawrence, Kansas 66044