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ARCS DEFINED BY ONE-PARAMETER SEMIGROUPS OF OPERATORS

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ABSTRACT. Let $T(t)$ $(t \geqq 0)$ be a one-parameter semigroup of continuous linear operators in a locally convex reflexive linear topological space X such that $T(c)$ is an isomorphism (into) for some $c>0$. It is proved that for any $x \in X$, $T(\cdot)x$ is of bounded variation on finite intervals if and only if x is in the domain of the infinitesimal generator of $T(t)$. The result is interpreted geometrically in terms of arc-length.

1. **Definitions and main results.** Let X be a locally convex sequentially complete (Hausdorff) linear topological space over the complex numbers, and let X^* and $L(X)$ denote the dual space and the space of all continuous linear operators in X respectively. A *one-parameter semigroup* is a family $\{T(t): t \geqq 0\} \subset L(X)$ with the following properties:

(1) $T(t)T(s) = T(t+s)$ for any t, $s \geqq 0$;

(2) $T(0) = I$, the identity operator;

(3) $\lim_{t \to s} T(t)x = T(s)x$ for any $s \geqq 0$ and any $x \in X$.

A one-parameter semigroup $\{T(t): t \geqq 0\} \subset L(X)$ is said to be *locally equicontinuous* if for any $0 < s < \infty$ and any continuous seminorm p on X there exists a continuous seminorm q on X such that $p(T(t)x) \leqq q(x)$ for all $0 \leqq t \leqq s$ and all $x \in X$. We refer the reader to the paper by T. Kōmura [2] for the basic properties of locally equicontinuous semigroups. In particular we note that a one-parameter semigroup is locally equicontinuous if the space X is reflexive.

The infinitesimal generator A of a one-parameter semigroup $\{T(t);\ t \geqq 0\}$ is defined by

$$Ax = \lim_{h \downarrow 0} (T(h)x - x)/h$$

whenever this limit exists. The domain of A, denoted $D(A)$, is always

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dense in X, and A is a linear operator from $D(A)$ into X. Furthermore, if $\{T(t):t\geqq 0\}$ is locally equicontinuous, then A is closed [2].

Let $x\in X$, $0\leqq a\leqq b<\infty$, and let p be any continuous seminorm on X. We define the *p-length* $L(x; a, b; p)$ of the arc $\{T(t)x:a\leqq t\leqq b\}$ by

$$L(x; a, b; p) = \sup \sum_{j=1}^{n} p(T(t_j)x - T(t_{j-1})x)$$

where the supremum is taken over all finite sums determined by partition points $a=t_0<t_1<\cdots<t_n=b$. If $L(x; a, b; p)<\infty$ for every continuous seminorm p, we say that the function $T(\cdot)x:[0, \infty)\to X$ is of *bounded variation* on the interval $[a, b]$. Finally $T(\cdot)x$ is said to be *absolutely continuous* on $[a, b]$ if for every positive ε and continuous seminorm p on X there exists a positive δ such that

$$\sum_{j=1}^{n} p(T(b_j)x - T(a_j)x) < \varepsilon$$

whenever $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ are nonoverlapping subintervals of $[a, b]$ with

$$\sum_{j=1}^{n} (b_j - a_j) < \delta.$$

We recall that an isomorphism of a locally convex space X into itself is a continuous linear transformation on X which is one-one and relatively open.

THEOREM. *Let X be a locally convex reflexive linear topological space and $\{T(t):t\geqq 0\}\subset L(X)$ a one-parameter semigroup with infinitesimal generator A. Suppose $T(c)$ is an isomorphism for some $c>0$. Then for each $x\in X$ the following are equivalent:*

(i) *$x\in D(A)$;*

(ii) *$T(\cdot)x$ is absolutely continuous on each finite subinterval $[a, b]$ of $[0, \infty)$;*

(iii) *$T(\cdot)x$ is of bounded variation on each finite subinterval $[a, b]$ of $[0, \infty)$.*

If any of the above statements holds then every arc $\{T(t)x:a\leqq t\leqq b\}$, where $0\leqq a<b<\infty$, has finite p-length, and

$$L(x; a, b; p) = \int_a^b p(AT(t)x)\, dt$$

for each continuous seminorm p on X.

COROLLARY. *If $D(A)\neq X$ (for example if X is an F-space and A is not continuous) then for each interval $[a, b]\subset[0, \infty)$ there exist continuous* *seminorms p on X such that each nonempty open set contains points x for which $L(x; a, b; p)$ is arbitrarily large (both finite and infinite).*

REMARKS. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) of the Theorem are valid in the absence of reflexivity. That this is *not* the case for the implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) is demonstrated by the following example:

EXAMPLE 1. Let $X=L^1(-\infty, \infty)$ and let $T(t)$ be translation by t. $D(A)$ consists of those $x \in X$ which are absolutely continuous on finite intervals and have derivatives in L^1, and $Ax=x'$ [**1**]. If $x(t)$ is the characteristic function of the interval $[0, 1]$, then $T(\cdot)x$ is absolutely continuous, but $x \notin D(A)$.

The next example shows that the implication (iii)$\Rightarrow$(i) of the Theorem is not generally true if none of the operators $T(t)$ $(t>0)$ is an isomorphism.

EXAMPLE 2. Let $X=L^2([0, \infty))$ and define $T(t)$ as in Example 1. Then $D(A)$ consists of all $x \in X$ which are absolutely continuous on finite intervals and have derivatives in L^2, and $Ax=x'$ [**1**]. Define $x \in X$ as follows:

$$\begin{aligned} x(s) &= s^{-1/4} && \text{if } 0 < s \leqq 1, \\ &= 2 - s && \text{if } 1 \leqq s \leqq 2, \\ &= 0 && \text{otherwise.} \end{aligned}$$

It is easily verified that $T(t)x \in D(A)$ for all $t>0$, but $x \notin D(A)$. Since

$$\|AT(t)x\| \leqq t^{-3/4} + 1 \quad \text{for all } t > 0,$$

it follows that for any partition $0 \leqq t_0 < t_1 < \cdots < t_n = b$ we have

$$\begin{aligned} \sum_{j=1}^{n} \|T(t_j)x - T(t_{j-1})x\| &\leqq 2\,\|x\| + \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \|AT(t)x\|\, dt \\ &\leqq 2\,\|x\| + b + 4b^{1/4}. \end{aligned}$$

Therefore $T(\cdot)x$ is of bounded variation on $[0, b]$ for every $0<b<\infty$. In fact, similar calculations show that $T(\cdot)x$ is absolutely continuous on $[a, \infty)$ for any $a>0$.

Our final example explicitly illustrates the phenomenon indicated in the Corollary.

EXAMPLE 3. Let $X=l^p$, $1 \leqq p < \infty$, and define $T(t)$ by

$$T(t)x = (e^{it}x_1, e^{i2t}x_2, \cdots); \qquad x = (x_1, x_2, \cdots).$$

Then $D(A)$ consists of all those $x=(x_1, x_2, \cdots) \in X$ such that

$$\sum_{n=1}^{\infty} |nx_n|^p < \infty,$$

and $Ax=(ix_1, 2ix_2, \cdots)$. Choose any $z=(z_1, z_2, \cdots)$ not in $D(A)$, and for each pair of positive integers k and n define $y(k, n)=(0, 0, \cdots, 0, z_k, z_{k+1}, \cdots, z_{k+n}, 0, 0, \cdots)$. Then $\lim_{k\to\infty} y(k, n)=0$ uniformly in n, and for each k,

$$L(y(k, n); a, b; \|\cdot\|) = (b-a)\,\|Ay(k, n)\| \to \infty$$

as $n\to\infty$. Thus for fixed $0\leqq a<b<\infty$ we may construct vectors y in $D(A)$ with arbitrarily small norm and arbitrarily large arc-length.

2. **Proof of Theorem.** Assume statement (i) of the Theorem holds. It follows from the sequential completeness of X and the continuity of $AT(\cdot)x$ on $[0, \infty)$ that the Riemann integral $\int_a^b AT(t)x\,dt$ exists in X for any $0\leqq a<b<\infty$. Moreover,

$$T(b)x - T(a)x = \int_a^b AT(t)x\,dt \tag{1}$$

[2]. Let p be any continuous seminorm on X and $0\leqq a<b<\infty$. By the local equicontinuity of $\{T(t); t\geqq 0\}$ there exists a continuous seminorm q on X such that $p(AT(t)x)\leqq q(AT(a)x)$ for all $a\leqq t\leqq b$. Therefore if $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ are nonoverlapping intervals in $[a, b]$, we have from (1)

$$\begin{aligned}\sum_{j=1}^{n} p(T(b_j)x - T(a_j)x) &\leqq \sum_{j=1}^{n}\int_{a_j}^{b_j} p(AT(t)x)\,dt\\ &\leqq \sum_{j=1}^{n}(b_j-a_j)q(AT(a)x),\end{aligned}$$

which implies that $T(\cdot)x$ is absolutely continuous on $[a, b]$.

The proof of the implication (ii)$\Rightarrow$(iii) is entirely similar to that of the classical case and is omitted. To prove the implication (iii)$\Rightarrow$(i) we shall need the following lemmas:

LEMMA 1. *Let X be a reflexive locally convex linear topological space and $\{T(t): t\geqq 0\}\subset L(X)$ a one-parameter semigroup. If $\{y_n\}$ is a bounded sequence in X, then there exists $y\in X$ such that for each $x^*\in X^*$ there exists a subsequence $\{x_n\}$ of $\{y_n\}$ for which*

$$\lim_{n\to\infty}\langle T(t)x_n - T(t)y, x^*\rangle = 0 \quad \text{for all } t\geqq 0.$$

PROOF. Since X is reflexive and $\{y_n\}$ is bounded, we may choose a weak cluster point $y\in X$ of $\{y_n\}$. Let $x^*\in X^*$ and let $r_1, r_2, \cdots$, be an enumeration of the nonnegative rationals. For each positive integer n let

$$U_n = \{x: |\langle y-x, T(r_j)^*x^*\rangle| < n^{-1}, 1\leqq j\leqq n\},$$

where $T(t)^*$ denotes the operator dual of $T(t)$. Since y is a weak cluster point of $\{y_n\}$, we may choose a subsequence $\{x_n\}$ of $\{y_n\}$ such that $x_n \in U_n$ for every n. Let B be any bounded set in X containing $\{y_n\}$. Then $q(y^*)=\sup\{|\langle x, y^*\rangle| : x \in B\}$ defines a continuous seminorm for the strong topology of X^*, so by the reflexivity of X it follows that (see **[2]**)

$$\lim_{r\to t} q(T(r)^*x^* - T(t)^*x^*) = 0.$$

Let $\varepsilon>0$, $t\geqq 0$, and choose r_j so that

$$\sup_n |\langle T(r_j)x_n - T(t)x_n, x^*\rangle| \leqq q(T(r_j)^*x^* - T(t)^*x^*) < \varepsilon$$

and

$$|\langle T(r_j)y - T(t)y, x^*\rangle| < \varepsilon.$$

If $n>\max\{j, 1/\varepsilon\}$, then

$$|\langle x_n - y, T(r_j)^*x^*\rangle| < 1/n < \varepsilon,$$

and therefore

$$\begin{aligned}|\langle T(t)x_n - T(t)y, x^*\rangle| &\leqq |\langle T(t)x_n - T(r_j)x_n, x^*\rangle| + |\langle T(r_j)(x_n - y), x^*\rangle| \\ &\quad + |\langle T(r_j)y - T(t)y, x^*\rangle| < 3\varepsilon,\end{aligned}$$

proving the lemma. Q.E.D.

LEMMA 2. *Let X be a reflexive locally convex linear topological space and $\{T(t):t\geqq 0\}\subset L(X)$ a one-parameter semigroup with infinitesimal generator A. Then $x \in D(A)$ if and only if $\{(T(h)x-x)/h : 0<h<1\}$ is a bounded subset of X.*

PROOF. The necessity is obvious. To prove the sufficiency we note that by Lemma 1 we may choose a point y in X such that given any $x^* \in X^*$, there exists a sequence

$$x_n = (T(h_n)x - x)/h_n, \qquad h_n \searrow 0,$$

such that for all $t\geqq 0$,

$$\lim_{n\to\infty} \langle T(t)x_n, x^*\rangle = \langle T(t)y, x^*\rangle.$$

Since $\{T(t):t\geqq 0\}$ is locally equicontinuous, given any positive r there exists a continuous seminorm p on X such that for all $z \in X$ and all $0\leqq t\leqq r$, $|\langle T(t)z, x^*\rangle| \leqq p(z)$. Therefore, since $\{x_n\}$ is bounded,

$$\sup\{|\langle T(t)x_n, x^*\rangle| : 0 \leqq t \leqq r; n = 1, 2, \cdots\} < \infty.$$

By Lebesgue's dominated convergence theorem then,

$$\lim_{n\to\infty} \int_0^r \langle T(t)x_n, x^*\rangle\, dt = \int_0^r \langle T(t)y, x^*\rangle\, dt.$$

Since

$$\int_0^r T(t)x_n\,dt = (1/h_n)\int_0^{h_n} (T(t+r)x - T(t)x)\,dt \to T(r)x - x,$$

it follows that

$$\langle T(r)x - x, x^*\rangle = \left\langle \int_0^r T(t)y\,dt, x^*\right\rangle.$$

Therefore, since x^* was arbitrary,

$$T(r)x - x = \int_0^r T(t)y\,dt,$$

which implies that $x \in D(A)$. Q.E.D.

LEMMA 3. *Let X be a locally convex linear topological space and $\{T(t): t \geqq 0\} \subset L(X)$ a locally equicontinuous one-parameter semigroup. Then for each continuous seminorm p on X and $0 \leqq a < b < \infty$ there exists a continuous seminorm q on X such that for all $x \in X$ and all $0 < h < b - a$,*

$$p(T(b+h)x - T(b)x)/h \leqq L(x; a, b; q)/(b - a - h).$$

PROOF. Let p, a, b be given as in statement of lemma, and let q be a continuous seminorm on X such that $p(T(t)x) \leqq q(x)$ for all $x \in X$ and all $0 \leqq t \leqq b$. If $0 < h < b - a$ and n denotes the greatest integer in $(b-a)/h$, then for any integer j such that $1 \leqq j \leqq n$ we have $0 \leqq b - a - jh + h \leqq b$, so

$$\begin{aligned} p(T(b+h)x - T(b)x) & \\ &= p(T(b-a-jh+h)(T(a+jh)x - T(a+jh-h)x)) \\ &\leqq q(T(a+jh)x - T(a+jh-h)x). \end{aligned}$$

Therefore,

$$\begin{aligned} np(T(b+h)x - T(b)x) &\leqq \sum_{j=1}^{n} q(T(a+jh)x - T(a+jh-h)x) \\ &\leqq L(x; a, b; q), \end{aligned}$$

and since $(b-a-h)/h < n$, the conclusion readily follows. Q.E.D.

Returning to the proof of the Theorem, let us assume that statement (iii) holds. Given any continuous seminorm s on X, there exists, by virtue of the local equicontinuity of $T(t)$ and the fact that $T(c)$ is an isomorphism, a continuous seminorm p such that $s(y) \leqq p(T(t)y)$ for all $y \in X$ and all $t \in [0, c]$. Hence, given any interval $[a, b] \subset [0, c]$ we obtain by Lemma 3 a continuous seminorm q such that

$$s(T(h)x - x) \cdot h^{-1} \leqq 2L(x; a, b; q) \cdot (b - a)^{-1}$$

for all $0<h<(b-a)/2$. Therefore $\{h^{-1}(T(h)x-x):0<h<1\}$ is a bounded set in X, hence $x \in D(A)$ by Lemma 2.

To prove the final assertion of the Theorem, assume conditions (i)–(iii) hold and let $0 \leqq a<b<\infty$. Let p be a continuous seminorm on X and choose a sequence of sums

$$S_m = \sum_{j=1}^{n(m)} p(T(t_{jm})x - T(t_{j-1\,m})x),$$

where $a=t_{0m}<t_{1m}<\cdots<t_{n(m)m}=b$, such that

$$\lim_{m\to\infty} S_m = L(x;a,b;p) \quad \text{and} \quad \lim_{m\to\infty} \max_j(t_{jm} - t_{j-1\,m}) = 0.$$

Define $f_m:[a,b]\to[0,\infty)$ by

$$f_m = \sum_{j=1}^{n(m)} p((T(t_{jm})x - T(t_{j-1\,m})x)/(t_{jm} - t_{j-1\,m}))\cdot \chi_{[t_{j-1\,m},t_{jm}]}$$

where χ_Y denotes the characteristic function of a set $Y\subset[a,b]$. If $t\in[a,b]$ is not equal to any partition point t_{jm}, then a standard calculation shows that $f_m(t)\to p(AT(t)x)$. Therefore $f_m\to p(AT(\cdot)x)$ a.e. in $[a,b]$. Also, the sequence $\{f_m\}$ is uniformly bounded a.e. For by the local equicontinuity of the semigroup there exists a continuous seminorm q on X such that if $t_{j-1\,m}<t<t_{jm}$

$$\begin{aligned} f_m(t) &= (t_{jm} - t_{j-1\,m})^{-1} p(T(t_{jm})x - T(t_{j-1\,m})x) \\ &\leqq (t_{jm} - t_{j-1\,m})^{-1}\int_{t_{j-1\,m}}^{t_{jm}} p(AT(s)x)\,ds \\ &\leqq q(AT(a)x). \end{aligned}$$

Therefore, by Lebesgue's dominated convergence theorem,

$$\int_a^b p(AT(t)x)\,dt = \lim_{m\to\infty}\int_a^b f_m(t)\,dt = \lim_{m\to\infty} S_m = L(x;a,b;p),$$

completing the proof of the Theorem.

3. **Proof of corollary.** We observe first that given a continuous seminorm q on X and an interval $[a,b]\subset[0,\infty)$, there exists for each integer $n\geqq 1$ a continuous seminorm p such that

$$L(x;a,b;q) \leqq L(x;a+nc,b+nc;p).$$

Hence it suffices to prove the Corollary for the case $[a,b]\subset[0,c]$.

Let U be a nonempty open subset of X. Since $D(A)\neq X$, there exists a point x in $U\backslash D(A)$. By the proof of the Theorem, there exists a continuous seminorm q such that $L(x; a, b; q) = +\infty$. Now define

$$x(r) = r^{-1}\int_0^r T(t)x\,dt \qquad (r > 0).$$

Then $x(r) \in D(A) \cap U$ for all sufficiently small r, and $Ax(r) = r^{-1}(T(r)x - x)$. By Lemma 2, there exists a continuous seminorm s such that $s(Ax(r))$ is arbitrarily large for small r. Choosing a continuous seminorm p such that $s(Ax(r)) \leqq p(T_t Ax(r))$ $(0 \leqq t \leqq c)$ and noting that

$$L(x(r); a, b; p) = \int_a^b p(T(t)Ax(r))\,dt \geqq (b - a)s(Ax(r)),$$

we see that $L(x(r); a, b; p)$ is arbitrarily large for small r.

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