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# ON SMALLEST COMPACTIFICATION FOR CONVERGENCE SPACES

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ABSTRACT. In this note we obtain necessary and sufficient conditions for a convergence space to have a smallest Hausdorff compactification and to have a smallest regular compactification.

**Introduction.** A Hausdorff convergence space as defined in [**1**] always has a Stone-Čech compactification which can be obtained by a slight modification of the result in [**3**]. But in general this need not be the largest Hausdorff compactification of the space, and in fact it has been pointed out in [**4**] that the number of distinct maximal Hausdorff compactifications can be quite large. In this note we define the notion of local compactness for a Hausdorff convergence space and show that a Hausdorff noncompact convergence space has a smallest Hausdorff compactification iff the space is locally compact. With a view to obtain a more satisfactory compactification theory for convergence spaces, Richardson and Kent have considered regular compactifications in [**4**] and have obtained a characterization of the class of convergence spaces for which regular compactifications exist and have shown that each such convergence space has a largest regular compactification. In this note such a convergence space is called an $R$-convergence space, and it has been shown that an $R$-convergence space has a smallest regular compactification iff its pre-topological modification is a locally compact topological space.

1. For terms and results about convergence spaces used in this paper, we refer to [**1**] and [**4**]. A convergence space $(S, q)$, where $q$ is the convergence structure will be denoted simply by $S$, and $q$-convergence and $q$-adherence points will be referred to as $S$-convergence and $S$-adherence points respectively. $\dot{x}$ will denote the principal ultrafilter generated by $\{x\}$. For a filter $\mathscr{F}$ on $T$, if its trace on a subset $S$ of $T$ exists, will be denoted by $\mathscr{F} \cap S$, and the filter generated by $\mathscr{F} \cap S$ on $T$ will be denoted by $[\mathscr{F} \cap S]$. $S$ will be called $T$-open if $S$ belongs to every filter on $T$ that $T$-converges

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to some point of $S$. If $T$ and $T'$ denote two convergence spaces consisting of the same set and convergence structures $q$ and $q'$ respectively, then $T \geqq T'$ will mean $q \geqq q'$. The same notation $\geqq$ will be used to compare two compactifications of a given space in the usual sense and $=$ will indicate equivalent compactifications. A convergence space $S$ is regular if it is regular in the sense defined in [**4**], and is therefore also Hausdorff.

1.1. We state a lemma which we will use often without explicitly mentioning it: *if $f$ and $g$ are two continuous functions from a convergence space $T_1$ to a Hausdorff convergence space $T_2$ such that $f$ and $g$ agree on a dense subset $S$ of $T_1$, then $f=g$.* The proof is simple and therefore omitted.

Let $S$ be a Hausdorff noncompact convergence space and $T=S\cup\{z\}$, where $z$ does not belong to $S$. We make $T$ a convergence space by defining a convergence structure on $T$ as: a filter $\mathscr{F}$ on $T$, $T$-converges to $x$ in $S$ iff $S$ belongs to $\mathscr{F}$ and $\mathscr{F} \cap S$ $S$-converges to $x$, and for a filter $\mathscr{F}$ on $T$, $\mathscr{F}$ $T$-converges to $z$ iff $\mathscr{F} \geqq \mathscr{G} \cap \dot{z}$, where $\mathscr{G}$ is any filter on $T$ such that $\mathscr{G} \cap S$ has no $S$-adherence points. It is easy to see that $(T, i)$ is a Hausdorff compactification of $S$, where $i$ is the inclusion map.

Now we give a definition of local compactness for a Hausdorff convergence space which coincides with the usual definition if the convergence space is a topological space, and the compactifications under consideration are Hausdorff topological compactifications.

1.2. Definition. A Hausdorff convergence space is locally compact iff it is open in each of its Hausdorff compactifications.

Theorem 1.3. *A Hausdorff noncompact convergence space has a smallest Hausdorff compactification iff it is locally compact.*

Proof. Let $S$ be a Hausdorff noncompact locally compact convergence space and $(T, i)$ be the Hausdorff compactification of $S$ as constructed above; we will show that $(T, i)$ is the smallest Hausdorff compactification of $S$. Let $(T', f)$ be an arbitrary Hausdorff compactification of $S$, then we define a function $h$ from $T'$ onto $T$ as:

$$
\begin{aligned}
h(y) &= i \circ f^{-1}(y), && \text{for } y \text{ in } f(S) \\
&= z, && \text{for } y \text{ in } T' - f(S).
\end{aligned}
$$

We will show that $h$ is continuous. If $\mathscr{F}$ is a filter on $T'$ which is $T'$-converging to $y$ in $f(S)$, then $S$ being locally compact, $f(S)$ belongs to $\mathscr{F}$ and hence $h(\mathscr{F})$ $T$-converges to $h(y)$. If $\mathscr{F}$ $T'$-converges to $y$ in $T'-f(S)$ and: (a) trace of $\mathscr{F}$ on $f(S)$ does not exist, then $T'-f(S)$ belongs to $\mathscr{F}$, and hence $h(\mathscr{F})=\dot{z}$ which $T$-converges to $z=h(y)$; (b) trace of $\mathscr{F}$ on $f(S)$ exists, then $\mathscr{F} \cap f(S)$ has no $f(S)$-adherence points and therefore $h(\mathscr{F} \cap f(S))=h(\mathscr{F}) \cap i(S)$ has no $i(S)$-adherence points, and hence

$h(\mathscr{F})$ $T$-converges to $z=h(y)$. Since $h \circ f=i$, uniqueness of $h$ follows from the above mentioned lemma and therefore $(T', f) \geqq (T, i)$.

Next, let $S$ be a Hausdorff noncompact convergence space having a smallest Hausdorff compactification. We first show that if $h$ is a continuous function from a Hausodrff convergence space $T$ onto an arbitrary convergence space such that $h$ restricted to a dense subset $S$ of $T$ is an isomorphism, then $h(T-S) \cap h(S)=\varnothing$. Let $S_1=h^{-1} \circ h(S)$; it is sufficient to show that $S_1=S$. Now $S \subset S_1 \subset T$, $S_1$ is Hausdorff, and $S$ is dense in $S_1$. Let $g$ be the inverse of the map $h$ restricted to $S$ and $g_1$ be $g(h|S_1)$, then $g_1$ restricted to $S$ is identity, and hence $g_1$ is identity from $S_1$ to $S_1$. Since $g_1(S_1) \subset S$, we have $S=S_1$. From here we find that if $(T_1, g_1)$ and $(T_2, g_2)$ are two Hausdorff compactifications of $S$, and $h$ is a continuous function from $T_1$ onto $T_2$ such that $h \circ g_1=g_2$, then $h(g_1(S))=g_2(S)$ and $h(T_1-g_1(S))=T_2-g_2(S)$. Since $(T, i)$ constructed above is a Hausdorff compactification of $S$, the above remarks imply that the smallest Hausdorff compactification of $S$ must be a one point compactification, and therefore $S$ is open in its smallest Hausdorff compactification. From this we see that $S$ is open in each of its Hausdorff compactifications and hence is a locally compact convergence space.

From the proof of the above theorem it is clear that $(T, i)$ is coarser than every Hausdorff compactification of $S$, in which $S$ is open, and therefore, in particular, is coarser than any finite point Hausdorff compactification of $S$. If some Hausdorff compactification of $S$ is coarser than $(T, i)$, then it must be a one point compactification and therefore finer than $(T, i)$, and hence, it can be proved, by using the above mentioned lemma, that this compactification is equivalent to $(T, i)$.

Let $S$ be a locally compact Hausdorff topological space. Let us denote by $(T', j)$ the one point Hausdorff topological compactification of $S$, and by $(T, i)$ the one point Hausdorff compactification (in the convergence sense) of $S$ as constructed above, then the above remark implies $(T', j) \geqq (T, i)$. Since $T'$ is a compact Hausdorff topological space, Proposition 1.1 in [2] implies $T'$ is minimal Hausdorff, and hence Proposition 1.9 in [2] implies $(T', j)=(T, i)$, because $T$ is a Hausdorff convergence space and $T$ and $T'$ are one point compactifications of $S$.

THEOREM 1.4. *If $S$ is a locally compact Hausdorff topological space, then $(T, i)$ is its one point Hausdorff topological compactification.*

2. As it has been pointed out in [4], not every regular convergence space has a regular compactification. A characterization has been obtained in [4] for regular convergence spaces having a regular compactification. Such a convergence space will be called an $R$-convergence space, and in this section $S$ will always denote an $R$-convergence space.

If $(P, g)$ is a regular compactification of $S$, then, as shown in [**4**], $(\pi P, g)$ is a Hausdorff topological compactification of the Tychonoff space $\pi S$.

Let $K$ denote the set of all regular convergence structures $\gamma$ on $P$ which satisfy the following two conditions: $\gamma$ coincides with $\pi P$ relative to ultrafilter convergence, and if $g(S)$ belongs to $\mathscr{F}$, then $\mathscr{F}$ $\gamma$-converges to $y$ in $g(S)$ iff $g^{-1}(\mathscr{F})$ $S$-converges to $g^{-1}(y)$. These two conditions are consistent because $S$ being an $R$-convergence space, every ultrafilter finer than the neighborhood filter at $x$ $S$-converges to $x$ for all $x$ in $S$ [**4**]. If $P_0$ and $P_1$ denote the convergence spaces consisting of the set $P$ equipped with the inf and sup of convergence structures in $K$ respectively, then it is easy to see that $(P_0, g)$ and $(P_1, g)$ are regular compactifications of $S$, $\pi P_0=\pi P_1=\pi P$ and $P_1 \geqq P \geqq P_0$. Conversely, if $P'$ is a regular convergence space such that $P_1 \geqq P' \geqq P_0$, then $\pi P_1=\pi P_0=\pi P'$ and $(P', g)$ is a regular compactification of $S$. For the rest of this section $(P_0, g)$ and $(P_1, g)$ will denote the regular compactifications of $S$ as obtained above for a given regular compactification $(P, g)$ of $S$.

If $(T, f)$ is a Hausdorff topological compactification of the Tychonoff space $\pi S$, then Richardson and Kent have obtained in [**4**] a regular compactification $(T_1, f)$ of $S$. By construction $T=\pi T_1$ and $T_1$ is the finest regular convergence space on the set $T$ satisfying the following two conditions: (a) $T_1$ coincides with $T$ relative to ultrafilter convergence; and (b) if $f(S)$ belongs to $\mathscr{F}$, then $\mathscr{F}$ $T_1$-converges to $y$ in $f(S)$ iff $f^{-1}(\mathscr{F})$ $S$-converges to $f^{-1}(y)$. This leads to the following

PROPOSITION 2.1. *If $(P, g)$ is a regular compactification of $S$ and $(T, f) \geqq (\pi P, g)$ as Hausdorff topological compactifications of $\pi S$, then $(T_1, f) \geqq (P, g)$ as regular compactifications of the convergence space $S$.*

PROOF. Let $h$ be the continuous function from $T$ onto $\pi P$ such that $h \circ f=g$; then we will show that $h$ is continuous from $T_1$ onto $P$. If not, there exists a filter $\mathscr{F}_1$ on $T$ and $x_1$ in $T$ such that $\mathscr{F}_1$ $T_1$-converges to $x_1$, but $h(\mathscr{F}_1)$ does not $P$-converge to $h(x_1)$. We denote by $T_1'$ a convergence space on the set $T$ having a convergence structure defined as: $T_1'$ coincides with $T_1$ for all $y \neq x_1$ in $T$, and $\mathscr{F}$ $T_1'$-converges to $x_1$ iff $\mathscr{F}$ $T_1$-converges to $x_1$ and $h(\mathscr{F})$ $P$-converges to $h(x_1)$. $T_1' > T_1$ implies $T_1'$ is Hausdorff and $T_1'$ coincides with $T_1$, and therefore with $T$ relative to ultrafilter convergence, and therefore $T_1'$ is regular. It can be seen that $f(S)$ belongs to $\mathscr{F}$; then $\mathscr{F}$ $T_1'$-converges to $y$ in $f(S)$ iff $f^{-1}(\mathscr{F})$ $S$-converges to $f^{-1}(y)$. Hence we get a contradiction, and the result follows.

From this we have the following result, which has been obtained in [**4**]; if $(T, f)$ is the Stone-Čech topological compactification of the Tychonoff space $\pi S$, then $(T_1, f)$ is the largest regular compactification of $S$.

If we denote by $T_0$ the convergence space consisting of the set $T$ equipped with the inf of all regular convergence structures $\gamma$ on $T$ which satisfy the conditions (a) and (b) above for $\gamma$ in place of $T_1$, then $(T_0, f)$ is a regular compactification of $S$ and $\pi T_1 = \pi T_0 = T$. This implies $T_0 \geqq T$. In fact it can be seen that $T_0$ is the finest convergence space on the set $T$ having the following properties: if $y$ does not belong to $f(S)$, then $\mathscr{V}_T(y)$ $T_0$-converges to $y$, where $\mathscr{V}_T(y)$ denotes the neighborhood filter in $T$ of the point $y$; if $y$ belongs to $f(S)$ and trace of $\mathscr{V}_T(y)$ on $T-f(S)$ exists, then $\mathrm{Cl}_T \mathscr{F} \cap [\mathscr{V}_T(y) \cap (T-f(S))]$ $T_0$-converges to $y$, where $f(S)$ belongs to $\mathscr{F}$ and $f^{-1}(\mathscr{F})$ $S$-converges to $f^{-1}(y)$; if $y$ belongs to $f(S)$ and trace of $\mathscr{V}_T(y)$ on $T-f(S)$ does not exist, then $\mathrm{Cl}_T \mathscr{F}$ $T_0$-converges to $y$, where $f(S)$ belongs to $\mathscr{F}$ and $f^{-1}(\mathscr{F})$ $S$-converges to $f^{-1}(y)$.

PROPOSITION 2.2. *If $(P, g)$ is a regular compactification of $S$ and $(\pi P, g) \geqq (T, f)$ as Hausdorff topological compactifications of the Tychonoff space $\pi S$, then $(P, g) \geqq (T_0, f)$ as regular compactifications of the convergence space $S$.*

PROOF. Let $h$ be the continuous function from $\pi P$ onto $T$ such that $h \circ g = f$. We will show that $h$ is continuous from $P$ onto $T_0$. Let $\mathscr{F}$ be a filter on $P$ which $P$-converges to some $y$ not in $f(S)$. Then by construction of $T_0$, $h(\mathscr{F})$ $T_0$-converges to $h(y)$. If $y$ belongs to $g(S)$ and $g(S)$ belongs to $\mathscr{F}$, then clearly $h(\mathscr{F})$ $T_0$-converges to $h(y)$. If $y$ belongs to $g(S)$ and $g(S)$ does not belong to $\mathscr{F}$, then let $\mathscr{G}$ denote the filter $\mathscr{F} \cap (P-g(S))$. Also, if $\mathscr{F}$ has a trace on $g(S)$, we denote it by $\mathscr{H}$. Then $\mathscr{F} = [\mathscr{G}] \cap [\mathscr{H}]$; now by construction of $T_0$, $h(\mathscr{F})$ $T_0$-converges to $h(y)$. Hence $h$ is continuous from $P$ onto $T_0$ and therefore the result follows.

For the rest of this section $(T_1, f)$ and $(T_0, f)$ will denote the regular compactifications of $S$ as obtained by the Richardson and Kent method and the above method, respectively, from a given Hausdorff topological compactification $(T, f)$ of the Tychonoff space $\pi S$.

If we write $(P, g) \sim (Q, h)$, where $(P, g)$ and $(Q, h)$ are two regular compactifications of $S$ if $(\pi P, g) = (\pi Q, h)$, that is $(\pi P, g)$ and $(\pi Q, h)$ are two equivalent topological compactifications of the Tychonoff space $\pi S$, then $\sim$ is an equivalence relation on the set of all regular compactifications of $S$.

THEOREM 2.3. *If $S$ is an $R$-convergence space, then each equivalence class of regular compactifications of $S$ has a largest and a smallest member.*

PROOF. Let $(P, g) \sim (Q, h)$. Since $P_1 \geqq P \geqq P_0$ and $Q_1 \geqq Q \geqq Q_0$, we need only show that $(P_1, g) = (Q_1, h)$ and $(P_0, g) = (Q_0, h)$. The first of these follows from Proposition 2.1 and the second from Proposition 2.2.

If $\pi S$ is a locally compact topological space and $(T, f)$ is the one point Hausdorff topological compactification of $\pi S$, then from Proposition 2.2 we find that $(T_0, f)$ is the smallest regular compactification of $S$. On the other hand, if $S$ has a smallest regular compactification $(P, g)$, then it is easy to verify that $(\pi P, g)$ is the smallest Hausdorff topological compactification of $\pi S$ and therefore $\pi S$ is a locally compact topological space. This leads to the following.

THEOREM 2.4. *If $S$ is an $R$-convergence space, then $S$ has a smallest regular compactification iff $\pi S$ is a locally compact topological space.*

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