Parallel Vector Fields and Periodic Orbits

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Abstract. Let \( V \) be a parallel vector field on a compact Riemannian manifold without boundary. Suppose the Euler class over the reals of the normal bundle to \( V \) is different from zero. Then the flow defined by \( V \) has a periodic orbit.

Let \( M^n \) be a \( C^\infty \) compact oriented \( n \)-dimensional Riemannian manifold, and let \( V \) be a nowhere vanishing \( C^\infty \) contravariant vector field on \( M^n \) that is parallel with respect to the metric; that is, we assume that the covariant derivative of \( V \) is zero. Let \( \mu \in H^{n-1}(M^n, R) \) be the Euler class of the normal bundle to \( V \) with real coefficients, where we put the obvious orientation on the normal bundle. The purpose of the note is to prove the following result:

**Theorem.** If \( \mu \neq 0 \), the flow defined by \( V \) has a periodic orbit.

Proof. We proceed by adopting a device used in [2]. The one-form \( \alpha \) gotten from \( V \) by lowering indices has covariant derivative zero and therefore has exterior derivative zero, since the exterior derivative is the skew-symmetrization of the covariant derivative. Moreover the interior product of \( \alpha \) and \( V \) is certainly never zero. Let \( \alpha_1, \cdots, \alpha_k \) be closed one-forms on \( M^n \) corresponding to a basis for the one-dimensional cohomology on \( M^n \). For some \( \delta > 0 \), \( |\varepsilon_1| + \cdots + |\varepsilon_k| < \delta \) implies that \( \alpha + \varepsilon_1 \alpha_1 + \cdots + \varepsilon_k \alpha_k \) has a nowhere vanishing interior product with \( V \). We see from this that we can get \( C^\infty \) one-forms \( \omega_1, \cdots, \omega_k \) each of which has a nonvanishing interior product with \( V \) and which correspond to a basis of the rational one-dimensional cohomology of \( M^n \). Then by multiplying \( \omega_1, \cdots, \omega_k \) by suitable rational constants we can get one-forms \( \omega'_1, \cdots, \omega'_k \) each of which has a nowhere vanishing interior product with \( V \) and which correspond to a basis of \( H'(M^n, R) \) arising from the integral one-dimensional cohomology of \( M^n \). Each of the one-forms \( \omega'_1, \cdots, \omega'_k \) arises from a map to the circle; in an easily understood notation there exist functions \( \theta_1, \cdots, \theta_k \) on \( M^n \) defined mod 1 such that \( \omega'_1 = d\theta_1, \cdots, \omega'_k = d\theta_k \).

Next we observe that if \( N_1, \cdots, N_k \) are the \((n-1)\)-dimensional manifolds corresponding to the equations \( \theta_1 = 0, \cdots, \theta_k = 0 \) and taken with...
the obvious orientation the fundamental class of $N_i$ yields by injection
the element of $H_{n-1}(M^n, R)$ which corresponds by Poincaré duality to
the cohomology class determined by $\omega_i$. This can be seen, for example,
by noticing that for any closed $(n-1)$-form $\lambda$, $\int_{M^n} \lambda \wedge d\theta_i = \int_{N_i} \lambda$, which
follows by using the local product structure on $M^n$ as a bundle over the
circle and noting that the integral of $\lambda$ is the same over each fibre for any
one of our fibrations $\theta_i$.

Since the Euler class $\mu$ of the normal bundle to the vector field $V$ is
assumed different from zero, there is an $i_0$ such that the cap product of $\mu$
with the injection of the fundamental class of $N_{i_0}$ into the homology of
$M^n$ is different from zero. Then the pullback of $\mu$ to the cohomology of
$N_{i_0}$ is different from zero. This pullback is, however, just the Euler class
of the oriented tangent bundle to $N_{i_0}$; thus the Euler characteristic of
$N_{i_0}$ is different from zero.

However $N_{i_0}$ is clearly a global cross-section to the flow determined by
the vector field $V$. If $h$ is the homeomorphism of $N_{i_0}$ onto itself deter-
mined by the flow we can conclude by a theorem of Fuller, since the Euler
characteristic of $N_{i_0}$ is different from zero, that there exists a point on $N_{i_0}$
periodic under $h$. Then the orbit of this point under the flow defined by
$V$ must be periodic.

(Note. After this paper was submitted, two related papers came to the
author's attention. In [2] Conley introduced the notion of a flow which
carries a one-form. A flow defined by a parallel vector field on a Rieman-
nian manifold carries a closed one-form. Moreover if the Euler class over
the reals of the normal bundle to a vector field $V$ on a compact orientable
manifold is different from zero, and if the flow defined by $V$ carries a
closed one-form, our argument can be carried over to prove that there
is a periodic orbit. In [1], which appeared after the present paper was
accepted for publication, Churchill shows that a flow which carries a
closed one-form has a cross-section.)

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