THE ERGODIC DECOMPOSITION OF CONSERVATIVE Baire MEASURES

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Abstract. Certain topological conditions on a Markov transition function are shown sufficient for an integral representation of conservative invariant Baire measures. The analysis incorporates the Choquet-Bishop-de Leeuw extension of the Krein-Milman theorem.

1. Introduction. A Markov process is prescribed here as a quartet \((X, \mathcal{B}, p(x, B), m)\), the components respectively a given state space, \(\sigma\)-algebra of events, Markov transition function, and any \(\sigma\)-finite initial measure satisfying the condition that all \(m\)-null events are invariant or equivalently that on \(\mathcal{B} : m(B) = 0 \Rightarrow m\{x : p(x, B) > 0\} = 0\). The requirement on \(m\) is the obverse of the usual notion of a nonsingular transition function and will be called \textit{presubinvariance}, a sine-qua-non for measures either invariant or conservative with respect to a given transition function.

The conservativeness of \(m\), for our purposes, is best defined following S. R. Foguel [4], as

\[
X = C(m) \iff \text{on } \mathcal{B} : \sum_{n=1}^{\infty} p^n(x, B) = \infty \text{ on } \mathcal{B}^e \text{ ae}(m),
\]

where \(\mathcal{B}^e\) is a minimal superset for \(B\) satisfying \(p(x, B) = 1_B \text{ ae}(m)\). When \(m\) is known to be conservative, \(\mathcal{B}\) can be replaced by the \(m\)-equivalent set \(\mathcal{B} = B \cup\{x : 0 < \sum_{n=1}^{\infty} p^n(x, B)\}\).

The invariance of \(m\) is defined using the operator \((\cdot)T\) on signed measures, whereby \(m\) is invariant (subinvariant) iff on \(\mathcal{B}\):

\[
mT(B) \equiv \int p(x, B) \, dm(x) = (\leq) \, m(B).
\]

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For invariant conservative \( m \), the \( \sigma \)-algebra of invariant events, \( \Sigma(m) \), is composed of sets \( B \) satisfying \( \tilde{T}1_B = 1_B \) \( ae(m) \), where \( \tilde{T}(\cdot) \) is an operator on \( L_\infty(m) \) defined by

\[
\tilde{T}(f) \equiv (d|dm)[(f^+ dm)T] - (d|dm)[(f^- dm)T],
\]

where \( f \) in \( L_\infty(m) \) is decomposed because \( f dm \) need not be a signed measure. Such invariant sets are identical with those more typically defined by the requirement that \( p(x, B) = 1_B \) \( ae(m) \), by [3, p. 77]. The operator \( \tilde{T} \) will prove useful when characterizing the geometry of ergodic \( m \).

From here on, \( X \) is at least a \( \sigma \)-compact, locally compact Hausdorff space, \( \mathcal{B} \) its Baire sets, and \( m \) is a Baire measure. \( (\mathcal{N}, \tau) \) denotes the collection of all Baire measures on \( \mathcal{B} \) with the weak topology \( \tau \) induced by the continuous functions with compact support, \( C_c(X) \). A net \( \{\nu_x\} \) in \( \mathcal{N} \) converges weakly to \( m \) in \( \mathcal{N} \) iff for all \( f \) in \( C_c(X) \): \( \int_X f d\nu_x \to \int_X f dm \).

Note that \( (\mathcal{N}, \tau) \) is a subspace of the locally convex topological vector space of linear, but not necessarily continuous, functionals on \( C_c(X) \).

\( X \) can be written as \( \bigcup_1^\infty K_n \) and \( \bigcup_1^\infty \emptyset_n \), where the \( K_n \) and \( \emptyset_n \) are, respectively, compact Baire sets and open bounded Baire sets such that \( \emptyset_n \subseteq K_n \subseteq \emptyset_{n+1} \subseteq K_{n+1} \). Subsequently, there is a useful realization of \( X \) as \( \bigcup_1^\infty E_n \) where the \( E_n \) are open bounded Baire sets identified as \( E_1 = \emptyset_1 \), \( E_2 = \emptyset_2 \), and for \( n \geq 3 \), \( E_n = \emptyset_n - K_{n-2} \). Any element of \( X \) lies in at most two of the \( E_n \). Using only those \( E_n \) for which \( m(E_n) > 0 \), the function

\[
u(x) \equiv \sum_n 1_{E_n}(x) \cdot [2^n m(E_n)]^{-1}
\]

lies in \( L_1(m) \cap L_\infty(m) \), and one can quickly show, for invariant \( m \), that the conservativeness of \( m \) can be written as

\[
X = C(m) \iff \text{for all } E_n \text{ such that } m(E_n) > 0: \sum_1^\infty p(x, E_n) = \infty \text{ } ae(m) \text{ on } E_n.
\]

In the following the notation \( E_n \) will refer to this particular realization of \( X \).

2. An integral representation. Let \( \mathcal{C} \) denote the conservative invariant elements of \( \mathcal{N} \). It is easily shown that \( \mathcal{C} \) is a positive convex cone. By definition, a presubinvariant \( m \) in \( \mathcal{N} \) is ergodic iff \( \Sigma(m) \) is trivial. The ray, \( \rho(m) \), generated by nonzero \( m \) in \( \mathcal{C} \) is extreme by definition iff convex representations of any element of \( \rho \) always involve only other elements of \( \rho \).

2.1. Proposition. A nonzero element \( v \) of \( \mathcal{C} \) is ergodic \( \iff \rho(v) \) is extreme.
Proof. Let \( v \) be ergodic with \( v = a \mu_1 + (1-a)\mu_2, \ a \in (0, 1) \), and \( \mu_1, \mu_2 \in \mathcal{C} \). Then \( \mu_1 \ll v, \mu_2 \ll v, \) and \( \mu_1 \) and \( \mu_2 \) cannot be mutually singular so that \( \mu_1 \perp \mu_2, \) where \( \lambda_1 \perp \mu_2 \) and \( \lambda_2 \perp \mu_1 \) and \( 0 \neq w_1 \ll \mu_2, 0 \neq w_2 \ll \mu_1 \).

Denoting \( d\mu_2/dv \) by \( f_2 \), we have \( 0 \leq f_2 \) is \( \mathcal{B} \)-measurable and finite \( \ae(v) \) on bounded sets. We can decompose \( X = \bigcup_1^\infty B_n \), the \( B_n \) disjoint bounded Baire sets, so that the functions \( g_k \) defined as \( (f_2 \wedge k) \cdot 1_{\bigcup_1^k B_n} \) on \( X \) and \( g_k \) is in \( L_1(v) \cap L_\infty(v) \). The operator \( (\cdot)T \) acting on \( L_1(v) \) via \( (g_k)T = (d/dv)(f_k dv)T \) is well extended, see [3, pp. 4–5], to act on \( \mathcal{B} \)-measurable positive functions by \( (f_2)T = \lim_k (f_k)T \). It follows from the invariance of \( v \) that \( f_2T = f_2 \ae(v) \), or since the \( g_k \) are in \( L_\infty(v) \) and since \( T(\cdot) \) acting on \( L_\infty(v) \) is precisely this extension of \( (\cdot)T \), that \( TF_2 = f_2 \ae(v) \). By [3, p. 76], \( v \) is conservative with respect to the adjoint of \( T \). So by [3, p. 21] \( f_2 \) is \( \Sigma(v) \)-measurable and there exists a constant \( c_2 \) such that \( \mu_2 = c_2v \). Similarly, \( d\mu_1/dv = c_1a(e(v)) \), and so \( \mu_1 \) and \( \mu_2 \) lie on \( \rho(v) \). The converse is quickly shown by contradiction and the proposition is proven.

Two nonzero ergodic elements of \( \mathcal{C} \) are either mutually singular or proportional. Referring to [6], if \( \mathcal{C} \) were weakly closed in \( \mathcal{N} \), we would have the notion of caps, \( \mathcal{H} \), in \( \mathcal{C} \) defined as \( \mathcal{H} = \{ \lambda \in \mathcal{C} : M(\lambda) \leq 1 \} \), using a gauge functional \( M \). The set \( \mathcal{H}_1 = \{ \lambda \in \mathcal{C} : M(\lambda) = 1 \} \) will be called the cap lid. From [5, p. 236], the extreme points, \( \text{Ex}(\mathcal{H}) \), of a cap in \( \mathcal{C} \) are precisely the vertex \( \{0\} \) and the elements in the cap lid located on extreme rays of \( \mathcal{C} \). So for any cap \( \mathcal{H} \), the set \( \mathcal{E}(\mathcal{H}_1) = \text{Ex}(\mathcal{H}) - \{0\} \) is exactly representative of all nonzero singular ergodic measures in the cone \( \mathcal{C} \).

### 2.2. Theorem (Integral Representation).

If the transition function \( p(x, B) \) satisfies conditions

(i) \( C \neq \{0\} \),

(ii) \( T(\cdot) : C_c(X) \to C_c(X), \) where \( Tf(x) \equiv \int f(y)p(x, dy) \),

(iii) for any open bounded Baire set \( \mathcal{O} \) such that \( \mathcal{O} \) is unbounded, the set \( \{ x \in C : \sum_1^\infty p^n(x, \mathcal{O}) < \infty \} \) is open, then every nontrivial \( m \in \mathcal{C} \) lies in some cap lid \( \mathcal{H}_1 \), and there is a probability measure \( p_m \) concentrated on \( \mathcal{E}(\mathcal{H}_1) \) so that for all \( B \) in \( \mathcal{B} \): \( m(B) = \int \delta(\mathcal{H}_1) \lambda(B) dp_m(\lambda) \).

Proof. By a theorem of P. Meyer, [5, p. 238] and [6, p. 95], there is a cap \( \mathcal{H}_m \) such that \( m \in \mathcal{H}_1 \) whenever \( \mathcal{C} \) is weakly closed in \( \mathcal{N} \). Under (ii), \( (\cdot)T \) acting on measures is an operator on \( \mathcal{N} \). Given a net \( \{v_\alpha\} \) in \( \mathcal{C} \), \( m^\tau \)-limit \( v_\alpha \) satisfies for all \( f \in C_c(X) : \int f d(v_\alpha T) = \int f d(v_\alpha) = \int Tf dv_\alpha \rightarrow x \int T fdm = \int f d(mT) \rightarrow v_\alpha T = v_\alpha \rightarrow^x mT \rightarrow mT = m \).

Thus weak limit points of \( \mathcal{C} \) are invariant. An example to follow shows that \( \mathcal{C} \) is generally not closed under (i) and (ii) in the sense that weak limit points may not be conservative. If \( m(\mathcal{E}_n) < \infty, \mathcal{E}_n \subseteq \Sigma(m) \Rightarrow \mathcal{E}_n \subseteq C(m) \). We therefore concern ourselves with those \( E_n \) for which \( m(\mathcal{E}_n) = \infty \), denoting a typical such \( E_n \).
in the remainder of this argument by $\mathcal{O}$ and the set $\{x \in \hat{\mathcal{O}}: \sum_{n=1}^{\infty} p^n(x, \mathcal{O}) = \infty\}$. By (iii), $\mathcal{O} - \hat{\mathcal{O}}_\infty$ is an open subset of $\mathcal{O}$ avoiding $\hat{\mathcal{O}}_\infty$. Suppose $m(\mathcal{O} - \hat{\mathcal{O}}_\infty) > 0$. Then there is a compact Baire $K \subset \mathcal{O} - \hat{\mathcal{O}}_\infty$, $f \in C_c(X)$ with $0 \leq f \leq 1$ and $f = 1$ on $K$ and $0$ on $(\mathcal{O} - \hat{\mathcal{O}}_\infty)^c$ such that $0 < m(K) \leq \int f \, dm$, while for all $\alpha$, $\int f \, d\nu_\alpha(\mathcal{O} - \hat{\mathcal{O}}_\infty) = 0$, a contradiction. Therefore, with the supplementary “recurrence” condition (iii), $m$ is conservative and $\mathcal{O}$ is weakly closed in $\mathcal{M}$. Applying the Choquet-Bishop-de Leeuw theorem as described by R. Phelps in [6, pp. 30–31], there is a probability measure $p_m$ on $(\mathcal{H}_m, \mathcal{S})$, where $\mathcal{S}$ is the $\sigma$-algebra generated by the Baire sets of $\mathcal{H}_m$ and the set $Ex(\mathcal{H}_m)$, so that $p_m(\mathcal{E}(\mathcal{H}_m)) = 1$ and $p_m$ represents $m$ in the sense that for any continuous affine function $\phi$ on $\mathcal{H}_m$, $\phi(m) = \int_{\mathcal{H}_m} \phi(\lambda) \, dp_m(\lambda)$. Let $K$ be any compact Baire set. There is a sequence $\{g_n\}$ in $C_c(X)$ such that $g_n \downarrow 1_K$ and a corresponding sequence $\{\phi_n\}$ of continuous $\mathcal{S}$-measurable affine functions on $\mathcal{H}_m$ such that $\phi_n(m) = \int_X g_n \, dm = \int_X g_n \, d\lambda$. For some $\hat{n}$, $\phi_{\hat{n}}(m) < \infty$ and so for each $\lambda$ and $q = 0, 1, 2, \ldots$ $\phi_{\hat{n}+q}(\lambda) \rightarrow q \lambda(K)$, so that

$$m(K) = \int_{\sigma(\mathcal{H}_m)} \lambda(K) \, dp_m(\lambda).$$

Referring to [1, p. 5], the class of compact Baire sets in $\mathcal{B}$ is a $\pi$-system. It is easily shown that the collection $\{B \in \mathcal{B}: \lambda(B) \text{ is } \mathcal{S}\text{-measurable and } m(B) = \int_{\sigma(\mathcal{H}_m)} \lambda(B) \, dp_m(\lambda)\}$ is a $\sigma$-system and so equal to $\mathcal{B}$, yielding the theorem.

A more motivated argument towards the conservativeness of $m$ is provided by defining the support of $m$ as in [9, p. 308]. Then under condition (ii), $\hat{\mathcal{O}}$ is open so that $\text{supp}(m) \cap \hat{\mathcal{O}}$ and $\hat{\mathcal{O}}_\infty$ are “similar” in that both sets are nonvoid and unbounded, both necessarily intersect $m$-positive open Baire subsets of $\hat{\mathcal{O}}$, and under (iii): $\text{supp}(m) \cap \mathcal{O} = \hat{\mathcal{O}}_\infty \cap \mathcal{O} = m(\mathcal{O} - \hat{\mathcal{O}}_\infty) = 0$.

We remark that (ii) implies that each individual set $\{x: p^n(x, \mathcal{O}) > 0\}$ is bounded, but does not in view of the following example imply that their union, or $\hat{\mathcal{O}}$ in particular, is bounded.

3. The necessity of the recurrence hypothesis. An example, suggested by M. Rosenblatt, shows given (i) and (ii) that a supplementary condition like (iii) is necessary. Let $X$ be the real line $\mathbb{R}$ with the usual topology and let $Z$ denote the integers. A continuous function $h(y)$ on $\mathbb{R}$ is defined as follows for $0 < p < \frac{3}{2} < q < 1$ and $p + q = 1$ for $y \leq 0$, $h(y) = q$; for $y \geq 0$, $h(y)$ is defined in piecewise fashion on the unit interval containing integers $i$ and $i+1$ as $q$ for $y = i$, $i+1$, as $p$ for $i+1/(i+2) \leq y \leq i+1 - 1/(i+2)$, as $q(1 - \lambda) + \lambda p$ for $y = i + \lambda(i+2)$ or for $y = i+1 - \lambda(i+2)$ with $0 < \lambda < 1$. Graphically, in each positive integral interval, $h(y)$ is an inverted trapezoid.
Then for \( y \) in \( \mathbb{R} \) and \( B \) in \( \mathcal{B} \), \( p(y, B) = h(y) \mathbb{1}_B \) \((y+1)+[1-h(y)] \mathbb{1}_B (y-1) \) is a Markov transition function. For fixed \( x \) in \([0, 1)\), denote \( S_x = \{i + x, i \in \mathbb{Z}\} \). The transition function \( p(i+x, A) \) on \((S_x, \mathcal{B} \cap S_x)\) corresponds to a random walk on \( S_x \), and for \( x \in (0, 1) \) a conservative invariant measure for the corresponding process in \((R, \mathcal{B})\) is given by

\[
v_x(x - i) = \frac{(p/q)^i(1 - h(x))}{q}, \quad v_x(x) = 1,
\]

\[
v_x(x + i) = \frac{[h(x)h(x + 1) \cdots h(x + i - 1)]}{[1 - h(x + 1)] \cdots [1 - h(x + i)] \cdots [1 - h(x + 1)] \cdots [1 - h(x + i - 1)] \cdots [1 - h(x)]} \quad \text{for } i = 1, 2, 3, \text{ etc.}
\]

So condition (i) is satisfied. As well, condition (ii) is fulfilled since for \( f \) in \( C_c(X) \),

\[
\int f(y)p(z, dy) = h(z)f(z + 1) + [1 - h(z)]f(z - 1).
\]

However, the net \( \{v_x\} \) in \( \mathcal{C} \) converges weakly as \( x \to 0 \) to the invariant measure \( m = \{(q/p)^i\}_{i \in \mathbb{Z}} \) which is not conservative since for \( p \neq q \) there are no recurrent states and so no conservative presubinvariant measures on \( Z \). So \( \mathcal{C} \) is not, in general, weakly closed under (i) and (ii) and for example need not have a compact base.

4. Finite measures. In [7, p. 100], Lemma 1 states that the space of regular Borel probability measures on a compact \( X \) is weak-star compact. We could substitute Baire sets for Borel sets in the argument to obtain the weak-star compactness of the space \( \mathcal{R} \) of Baire probabilities on \( X \). Let \( \mathcal{P} \) be the convex subspace of invariant (and so conservative) Baire probabilities. Condition (ii), which now reads \( T:C(X) \to C(X) \), ensures that \( \mathcal{P} \) is weakly closed in \( \mathcal{R} \). \( \mathcal{P} \) is subsequently weakly compact since the weak and weak-star topologies inherited by \( \mathcal{R} \) have the same subbasis with sets of the form \( \{v \in \mathcal{R} : |\int f \, dv - \int f \, dv_0| < \epsilon\} \). Condition (ii) ensures \( \mathcal{P} \neq \{0\} \), from [7, p. 101]. Another result in [8] says that \( \text{Ex}(\mathcal{P}) \) consists precisely of ergodic elements of \( \mathcal{P} \) since ergodicity there can be shown to be equivalent to ours. So with little effort we have:

4.1. Proposition. Let \( X \) be compact. Then the condition \( T(\cdot): C(X) \to C(X) \) suffices to ensure that the set of invariant Baire probabilities, \( \mathcal{P} \), is nonvoid and weakly compact and that for any \( \mu \) in \( \mathcal{P} \) there exists a probability measure \( q_\mu \) so that for all \( B \) in \( \mathcal{B} \): \( \mu(B) = \int \delta(\mathcal{P}) \lambda(B) \, dq_\mu(\lambda) \), where \( \delta(\mathcal{P}) \) is the set of ergodic elements of \( \mathcal{P} \).

5. Transformation invariance. The invariance of finite measures is typically defined in representation oriented papers with respect to one or more measurable, measure-preserving transformations on \( X \). See [2] for a history of such efforts.
In [6, Chapter 10], there is an application of the Choquet-et al. theorem for transformation-invariant probabilities which is convenient for comparison. Let $X$ be compact and $\mathcal{P}$ be the collection of Baire probabilities invariant with respect to $\rho(x, B)$. Form the cartesian product $Y \equiv X^Z$ and the associated product $\sigma$-algebra $\mathcal{A} \equiv \mathcal{B}^Z$ which in fact, [5, p. 23], is the $\sigma$-algebra of Baire sets of $Y$. Denote the shift transformation-invariant measures by $\mathcal{P}_0$. There is an embedding $Y: \mathcal{P}_0 \rightarrow \mathcal{P}_\mu$ via $\mu \mapsto P_\mu$ where $P_\mu$ is the Carathéodory extension of the measure induced on any finite $X^F, F \equiv \{j_1, \cdots, j_f\}$ by

$$P_{\mu,F}(y \in X^F : y_{j_i} \in B_i, j_i \in F) = \text{prob}(x_0 \in B_1, x_1 \in B_2, \cdots, x_{f-1} \in B_f).$$

Moreover, using prediction theory, as in [7, p. 97] it can be shown that $\Gamma$ preserves ergodicity. If $\Gamma$ were surjective, one would then quickly subsume existing representations for shift transformation-invariant finite measures. But this is not the case. The process $\{y_j\} \text{ associated with } P_\mu$ is Markovian and stationary. For a given $X$, one need only produce a non-Markovian stationary process to exhibit a shift-invariant $P$ which is not the image under $\Gamma$ of any $\mu$ in $\mathcal{P}$.

References