ON \( \Delta(x, n) = \varphi(x, n) - x\varphi(n)/n \)

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Abstract. Let \( \Delta(x, n) = \varphi(x, n) - x\varphi(n)/n \), where \( \varphi(x, n) \) denotes the number of positive integers \( \leq x \) and prime to \( n \), \( \varphi(n) = \varphi(n, n) \). In this paper, lower and upper bounds for \( \Delta(x, n) \), which hold for all values of \( x \geq 1 \) and \( n \geq 2 \), are established.

1. Introduction. Throughout the following \( x \) denotes a real variable \( \geq 1 \) and \( n \) denotes a positive integer. \([x]\), denotes the greatest integer not exceeding \( x \) (integral part of \( x \)) and \( \{x\} = x - [x] \) (fractional part of \( x \)). Let \( \varphi(x, n) \) denote the Legendre totient function defined to be the number of positive integers \( \leq x \) which are prime to \( n \), and let \( \varphi(n) \) denote the Euler totient function defined to be the number of positive integers \( \leq n \) which are prime to \( n \), so that \( \varphi(n, n) = \varphi(n) \). It is well known (cf. [6, pp. 234–235]),

\[
\varphi(n) = n \sum_{d|n} \mu(d)/d = n \prod_{p|n} (1 - 1/p),
\]

where \( \mu(n) \) denotes the Möbius function and the product is extended over all prime divisors \( p \) of \( n \).

Let \( \psi(n) \) denote the Dedekind \( \psi \)-function (cf. [3, p. 123], also cf. [2]) defined by

\[
\psi(n) = n \sum_{d|n} \mu^2(d)/d = n \prod_{p|n} (1 + 1/p).
\]

It is easy (using Theorem 261 of [6], also cf. [3, p. 115]) to show that

\[
\varphi(x, n) = \sum_{d|n} \mu(d)\lfloor x/d \rfloor,
\]

from which it follows (using \( [x] = x + O(1) \)) that

\[
\varphi(x, n) = x\varphi(n)/n + O\left( \sum_{d|n} \mu^2(d) \right) = x\varphi(n)/n + O(\theta(n)),
\]

where \( \theta(n) \) denotes the number of square-free divisors of \( n \). The relation (1.4) may also be found in E. Cohen [1, Lemma 3.4].

Received by the editors July 6, 1973.


Key words and phrases. Legendre and Euler totient functions, Dedekind's \( \psi \)-function, the number of square-free divisors of \( n \).

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Throughout this note we write

\[(1.5) \quad \Delta(x, n) = \varphi(x, n) - x\varphi(n)/n.\]

Relation (1.4) implies that \(\Delta(x, n) = O(\theta(n))\). The object of the present note is to establish lower and upper bounds for \(\Delta(x, n)\) which hold for all values of \(x \geq 1\) and \(n \geq 2\). In fact, we prove the following:

**Theorem 1.** For all values of \(x \geq 1\) and \(n \geq 2\), we have

\[|\Delta(x, n) + (\{x\} - \frac{1}{2})\varphi(n)/n| \leq \frac{1}{2}(\theta(n) - \varphi(n)/n).\]

**Theorem 2.** For all values of \(x \geq 1\) and \(n \geq 2\) such that \([x], n = 1\), we have

\[
\frac{\varphi(n)}{n} - \frac{\theta(n)}{2} \leq \Delta(x, n) + \{x\} \frac{\varphi(n)}{n} \leq \frac{\theta(n)}{2} - \frac{\varphi(n)}{n} + 1.
\]

For a discussion of some of the results about \(\Delta(x, n)\) we refer to P. Erdös [4], [5], D. H. Lehmer [7, Lemma 4], and T. Vijayaraghavan [9].

2. **Proof of Theorem 1.** It is well known (cf. [8, Theorem 23]) that \([x]/n = [x/n]\), so that we have

\[
\frac{[x]}{n} - \left(\frac{[x]}{n}\right) = \frac{x}{n} - \frac{\{x\}}{n},
\]

and hence

\[
\frac{x}{n} - \frac{\{x\}}{n} - \left(\frac{[x]}{n}\right) = \frac{x}{n} - \frac{\{x\}}{n},
\]

which implies that

\[
\frac{\{x\}}{n} = \frac{\{x\}}{n} + \left(\frac{[x]}{n}\right).
\]

If we write \(f(x, n) = [x]/n\), we find that

\[(2.1) \quad \{x/n\} = \{x\}/n + f(x, n),\]

where

\[(2.2) \quad 0 \leq f(x, n) \leq 1 - 1/n.\]

We have, from (1.3) and (1.1),

\[
\varphi(x, n) = \sum_{d \mid n} \mu(d)\lfloor x/d \rfloor = \sum_{d \mid n} \mu(d)\left(\frac{x}{d} - \frac{\lfloor x\rfloor}{d}\right)
= x\sum_{d \mid n} \mu(d)/d - \sum_{d \mid n} \mu(d)\{x/d\}
= x\varphi(n)/n - \sum_{d \mid n} \mu(d)\{x/d\},
\]

so that, from (1.5) and (2.1),

\[ \Delta(x, n) = -\sum_{d|n} \mu(d)\{x/d\} = -\sum_{d|n} \mu(d)\{x\}/d - \sum_{d|n} \mu(d)f(x, d) \]

\[ = -\{x\} \varphi(n)/n - \sum_{d|n} \mu(d)f(x, d). \]

Hence

\[ \Delta(x, n) + \{x\} \varphi(n)/n = -\sum_{d|n} \mu(d)f(x, d) \]

(2.3)

\[ = \sum_{d|n: \omega(d) \text{ is odd}} \mu^2(d)f(x, d) - \sum_{d|n: \omega(d) \text{ is even}} \mu^2(d)f(x, d), \]

where \( \omega(n) \) is the number of distinct prime factors of \( n > 1 \). Since \( f(x, 1) = 0 \) and \( f(x, d) \geq 0 \), we have, from (2.3) and (2.2),

\[ \Delta(x, n) + \{x\} \varphi(n)/n \leq \sum_{d|n: \omega(d) \text{ is odd}} \mu^2(d)f(x, d) \]

(2.4)

\[ \leq \sum_{d|n: \omega(d) \text{ is odd}} \mu^2(d)(1 - (1/d)) \]

\[ = \sum_{d|n: \omega(d) \text{ is odd}} \mu^2(d) - \sum_{d|n: \omega(d) \text{ is even}} \mu^2(d)/d. \]

But

\[ \sum_{d|n: \omega(d) \text{ is odd}} \mu^2(d) = \left( \frac{\omega(n)}{1} \right) + \left( \frac{\omega(n)}{3} \right) + \cdots = 2^{\omega(n) - 1} = \theta(n)/2, \]

and, from (1.1) and (1.2),

\[ \sum_{d|n: \omega(d) \text{ is odd}} \mu^2(d)/d = \frac{1}{2} \left( \frac{\psi(n)}{n} - \varphi(n)/n \right). \]

Hence from (2.4), we have

\[ \Delta(x, n) + \{x\} \varphi(n)/n \leq \frac{1}{2} \left( \theta(n) - \psi(n)/n + \varphi(n)/n \right), \]

so that

(2.5) \[ \Delta(x, n) + \{x\} - \frac{1}{2})\varphi(n)/n \leq \frac{1}{2} \left( \theta(n) - \psi(n)/n \right). \]

Also from (2.3) and (2.2), we have

\[ \Delta(x, n) + \{x\} \varphi(n)/n \geq -\sum_{d|n: \omega(d) \text{ is even}} \mu^2(d)f(x, d) \]

(2.6)

\[ \geq -\sum_{d|n: \omega(d) \text{ is even}} \mu^2(d)(1 - 1/d) \]

\[ = -\sum_{d|n: \omega(d) \text{ is even}} \mu^2(d) + \sum_{d|n: \omega(d) \text{ is even}} \mu^2(d)/d. \]
But
\[ \sum_{d \mid n : \omega(d) \text{ is even}} \mu^2(d) = \left( \frac{\omega(n)}{2} \right) + \left( \frac{\omega(n)}{4} \right) + \cdots = 2^{\omega(n)-1} - 1 \]
= \theta(n)/2 - 1,
and from (1.1) and (1.2),
\[ \sum_{d \mid n : \omega(d) \text{ is even}} \mu^2(d)/d = \frac{1}{2} \left( \frac{\psi(n)}{n} + \frac{\varphi(n)}{n} \right) - 1. \]
Hence from (2.6), we have
\[ \Delta(x, n) + \{x\} \varphi(n)/n \geq \frac{1}{2} \left( \frac{\theta(n)}{2} - 1 \right) + \frac{1}{2} \left( \frac{\psi(n)}{n} + \frac{\varphi(n)}{n} \right) - 1, \]
so that
\[ \Delta(x, n) + \{x\} \varphi(n)/n \geq \frac{1}{2} \left( \psi(n)/n - \theta(n) \right). \]
Now, Theorem 1 follows from (2.5) and (2.7).

3. Proof of Theorem 2. Since ([x], n)=1, we have \( f(x, n) = \{[x]/n\} \leq 1/n \), so that in this case, we have
\[ 1/n \leq f(x, n) \leq 1 - (1/n). \]
Hence, from (2.3) and (3.1), we have
\[ \Delta(x, n) + \{x\} \varphi(n)/n \]
\[ \leq \sum_{d \mid n : \omega(d) \text{ is odd}} \mu^2(d)(1 - 1/d) - \sum_{d \mid n : \omega(d) \text{ is even}} \mu^2(d)/d \]
= \sum_{d \mid n : \omega(d) \text{ is odd}} \mu^2(d) - \left( \sum_{d \mid n} \frac{\mu^2(d)}{d} - 1 \right)
= \frac{\theta(n)}{2} - \frac{\psi(n)}{n} + 1.
Also, from (2.3) and (3.1), we have
\[ \Delta(x, n) + \{x\} \varphi(n)/n \]
\[ \geq \sum_{d \mid n : \omega(d) \text{ is odd}} \mu^2(d)/d - \sum_{d \mid n : \omega(d) \text{ is even}} \mu^2(d)(1 - 1/d) \]
= \left( \sum_{d \mid n} \frac{\mu^2(d)}{d} - 1 \right) - \sum_{d \mid n : \omega(d) \text{ is even}} \mu^2(d)
= (\psi(n)/n - 1) - (\theta(n)/2 - 1)
= \frac{\psi(n)}{n} - \frac{\theta(n)}{2}.
Now, Theorem 2 follows from (3.2) and (3.3).
ON \( \Delta(x, n) = \varphi(x, n) - x\varphi(n)/n \)

REFERENCES


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