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ON $\Delta(x, n) = \varphi(x, n) - x\varphi(n)/n$

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ABSTRACT. Let $\Delta(x, n)=\varphi(x, n)-x\varphi(n)/n$, where $\varphi(x, n)$ denotes the number of positive integers $\leqq x$ and prime to n, $\varphi(n)=\varphi(n, n)$. In this paper, lower and upper bounds for $\Delta(x, n)$, which hold for all values of $x\geqq 1$ and $n\geqq 2$, are established.

1. **Introduction.** Throughout the following x denotes a real variable $\geqq 1$ and n denotes a positive integer. $[x]$ denotes the greatest integer not exceeding x (integral part of x) and $\{x\}=x-[x]$ (fractional part of x). Let $\varphi(x, n)$ denote the Legendre totient function defined to be the number of positive integers $\leqq x$ which are prime to n, and let $\varphi(n)$ denote the Euler totient function defined to be the number of positive integers $\leqq n$ which are prime to n, so that $\varphi(n, n)=\varphi(n)$. It is well known (cf. [**6**, pp. 234–235]), that

$$\varphi(n) = n \sum_{d|n} \mu(d)/d = n \prod_{p|n} (1 - 1/p), \tag{1.1}$$

where $\mu(n)$ denotes the Möbius function and the product is extended over all prime divisors p of n.

Let $\psi(n)$ denote the Dedekind ψ-function (cf. [**3**, p. 123], also cf. [**2**]) defined by

$$\psi(n) = n \sum_{d|n} \mu^2(d)/d = n \prod_{p|n} (1 + 1/p). \tag{1.2}$$

It is easy (using Theorem 261 of [**6**], also cf. [**3**, p. 115]) to show that

$$\varphi(x, n) = \sum_{d|n} \mu(d)[x/d], \tag{1.3}$$

from which it follows (using $[x]=x+O(1)$) that

$$\varphi(x, n) = x\varphi(n)/n + O\left(\sum_{d|n} \mu^2(d)\right) = x\varphi(n)/n + O(\theta(n)), \tag{1.4}$$

where $\theta(n)$ denotes the number of square-free divisors of n. The relation (1.4) may also be found in E. Cohen [**1**, Lemma 3.4].

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Throughout this note we write

$$\Delta(x, n) = \varphi(x, n) - x\varphi(n)/n. \tag{1.5}$$

Relation (1.4) implies that $\Delta(x, n)=O(\theta(n))$. The object of the present note is to establish lower and upper bounds for $\Delta(x, n)$ which hold for all values of $x\geqq 1$ and $n\geqq 2$. In fact, we prove the following:

THEOREM 1. *For all values of* $x\geqq 1$ *and* $n\geqq 2$, *we have*

$$|\Delta(x, n) + (\{x\} - \tfrac{1}{2})\varphi(n)/n| \leqq \tfrac{1}{2}(\theta(n) - \psi(n)/n).$$

THEOREM 2. *For all values of* $x\geqq 1$ *and* $n\geqq 2$ *such that* $([x], n)=1$, *we have*

$$\frac{\psi(n)}{n} - \frac{\theta(n)}{2} \leqq \Delta(x, n) + \{x\}\frac{\varphi(n)}{n} \leqq \frac{\theta(n)}{2} - \frac{\psi(n)}{n} + 1.$$

For a discussion of some of the results about $\Delta(x, n)$ we refer to P. Erdös [**4**], [**5**], D. H. Lehmer [**7**, Lemma 4], and T. Vijayaraghavan [**9**].

2. **Proof of Theorem 1.** It is well known (cf. [**8**, Theorem 23]) that $[[x]/n]=[x/n]$, so that we have

$$\frac{[x]}{n} - \left\{\frac{[x]}{n}\right\} = \frac{x}{n} - \left\{\frac{x}{n}\right\},$$

and hence

$$\frac{x}{n} - \frac{\{x\}}{n} - \left\{\frac{[x]}{n}\right\} = \frac{x}{n} - \left\{\frac{x}{n}\right\},$$

which implies that

$$\left\{\frac{x}{n}\right\} = \frac{\{x\}}{n} + \left\{\frac{[x]}{n}\right\}.$$

If we write $f(x, n)=\{[x]/n\}$, we find that

$$\{x/n\} = \{x\}/n + f(x, n), \tag{2.1}$$

where

$$0 \leqq f(x, n) \leqq 1 - 1/n. \tag{2.2}$$

We have, from (1.3) and (1.1),

$$\begin{aligned}\varphi(x, n) &= \sum_{d|n} \mu(d)[x/d] = \sum_{d|n} \mu(d)\left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right)\\ &= x\sum_{d|n} \mu(d)/d - \sum_{d|n} \mu(d)\{x/d\}\\ &= x\varphi(n)/n - \sum_{d|n} \mu(d)\{x/d\},\end{aligned}$$

so that, from (1.5) and (2.1),

$$\begin{aligned}\Delta(x, n) &= -\sum_{d|n} \mu(d)\{x/d\} = -\sum_{d|n} \mu(d)\{x\}/d - \sum_{d|n} \mu(d) f(x, d) \\ &= -\{x\}\varphi(n)/n - \sum_{d|n} \mu(d) f(x, d).\end{aligned}$$

Hence

$$\begin{aligned}\Delta(x, n) + \{x\}\varphi(n)/n &= -\sum_{d|n} \mu(d) f(x, d) \\ &= \sum_{d|n;\omega(d) \text{ is odd}} \mu^2(d) f(x, d) - \sum_{d|n;\omega(d) \text{ is even}} \mu^2(d) f(x, d),\end{aligned} \tag{2.3}$$

where $\omega(n)$ is the number of distinct prime factors of $n>1$. Since $f(x, 1)=0$ and $f(x, d) \geqq 0$, we have, from (2.3) and (2.2),

$$\begin{aligned}\Delta(x, n) + \{x\}\varphi(n)/n &\leqq \sum_{d|n;\omega(d) \text{ is odd}} \mu^2(d) f(x, d) \\ &\leqq \sum_{d|n;\omega(d) \text{ is odd}} \mu^2(d)(1 - (1/d)) \\ &= \sum_{d|n;\ \omega(d) \text{ is odd}} \mu^2(d) - \sum_{d|n;\omega(d) \text{ is odd}} \mu^2(d)/d.\end{aligned} \tag{2.4}$$

But

$$\sum_{d|n;\omega(d) \text{ is odd}} \mu^2(d) = \binom{\omega(n)}{1} + \binom{\omega(n)}{3} + \cdots = 2^{\omega(n)-1} = \theta(n)/2,$$

and, from (1.1) and (1.2),

$$\sum_{d|n;\ \omega(d) \text{ is odd}} \mu^2(d)/d = \frac{1}{2}\left(\frac{\psi(n)}{n} - \frac{\varphi(n)}{n}\right).$$

Hence from (2.4), we have

$$\Delta(x, n) + \{x\}\varphi(n)/n \leqq \frac{1}{2}\left(\theta(n) - \frac{\psi(n)}{n} + \frac{\varphi(n)}{n}\right),$$

so that

$$\Delta(x, n) + (\{x\} - \tfrac{1}{2})\varphi(n)/n \leqq \tfrac{1}{2}(\theta(n) - \psi(n)/n). \tag{2.5}$$

Also from (2.3) and (2.2), we have

$$\begin{aligned}\Delta(x, n) + \{x\}\varphi(n)/n &\geqq -\sum_{d|n;\omega(d) \text{ is even}} \mu^2(d) f(x, d) \\ &\geqq -\sum_{d|n;\omega(d) \text{ is even}} \mu^2(d)(1 - 1/d) \\ &= -\sum_{d|n;\omega(d) \text{ is even}} \mu^2(d) + \sum_{d|n;\omega(d) \text{ is even}} \mu^2(d)/d.\end{aligned} \tag{2.6}$$

But

$$\sum_{d|n;\omega(d)\text{ is even}} \mu^2(d) = \binom{\omega(n)}{2} + \binom{\omega(n)}{4} + \cdots = 2^{\omega(n)-1} - 1$$
$$= \theta(n)/2 - 1,$$

and from (1.1) and (1.2),

$$\sum_{d|n;\omega(d)\text{ is even}} \mu^2(d)/d = \frac{1}{2}\left(\frac{\psi(n)}{n} + \frac{\varphi(n)}{n}\right) - 1.$$

Hence from (2.6), we have

$$\Delta(x, n) + \{x\}\varphi(n)/n \geqq -\left(\frac{\theta(n)}{2} - 1\right) + \frac{1}{2}\left(\frac{\psi(n)}{n} + \frac{\varphi(n)}{n}\right) - 1,$$

so that

$$\Delta(x, n) + (\{x\} - \tfrac{1}{2})\varphi(n)/n \geqq \tfrac{1}{2}(\psi(n)/n - \theta(n)). \tag{2.7}$$

Now, Theorem 1 follows from (2.5) and (2.7).

3. **Proof of Theorem 2.** Since $([x], n)=1$, we have $f(x, n)=\{[x]/n\}\geqq 1/n$, so that in this case, we have

$$1/n \leqq f(x, n) \leqq 1 - (1/n). \tag{3.1}$$

Hence, from (2.3) and (3.1), we have

$$\begin{aligned}\Delta(x, n) &+ \{x\}\varphi(n)/n \\ &\leqq \sum_{d|n;\omega(d)\text{ is odd}} \mu^2(d)(1 - 1/d) - \sum_{d|n;\omega(d)\text{ is even}} \mu^2(d)/d \\ &= \sum_{d|n;\omega(d)\text{ is odd}} \mu^2(d) - \left(\sum_{d|n} \frac{\mu^2(d)}{d} - 1\right) \\ &= \frac{\theta(n)}{2} - \frac{\psi(n)}{n} + 1.\end{aligned} \tag{3.2}$$

Also, from (2.3) and (3.1), we have

$$\begin{aligned}\Delta(x, n) &+ \{x\}\varphi(n)/n \\ &\geqq \sum_{d|n;\omega(d)\text{ is odd}} \mu^2(d)/d - \sum_{d|n;\omega(d)\text{ is even}} \mu^2(d)(1 - 1/d) \\ &= \left(\sum_{d|n} \frac{\mu^2(d)}{d} - 1\right) - \sum_{d|n;\omega(d)\text{ is even}} \mu^2(d) \\ &= (\psi(n)/n - 1) - (\theta(n)/2 - 1) \\ &= \frac{\psi(n)}{n} - \frac{\theta(n)}{2}.\end{aligned} \tag{3.3}$$

Now, Theorem 2 follows from (3.2) and (3.3).

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