SUMS OF QUOTIENTS OF ADDITIVE FUNCTIONS

JEAN-MARIE DE KONINCK

Abstract. Denote by \( \omega(n) \) and \( \Omega(n) \) the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \), respectively. Given any positive integer \( a \), we prove that

\[
\sum_{2 \leq n \leq x} \Omega(n)/\omega(n) = x + x \sum_{i=1}^{a} a_i/(\log \log x)^i + O(x/(\log \log x)^{a+1}),
\]

where \( a_1 = \sum \frac{1}{p(p-1)} \) and all the other \( a_i \)'s are computable constants. This improves a previous result of R. L. Duncan.

Denote by \( \omega(n) \) and \( \Omega(n) \) the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \), respectively. R. L. Duncan [3] proved that

\[
\sum_{2 \leq n \leq x} \Omega(n)/\omega(n) = x + O(x/\log \log x).
\]

Duncan's result was based on the elementary estimate

\[
\sum_{2 \leq n \leq x} 1/\omega(n) = O(x/\log \log x).
\]

(1)

In a previous paper [1], we gave estimates of \( \sum_{n \leq x} 1/f(n) \) for a large class of additive functions \( f(n) \) (where \( \sum' \) denotes summation over those values of \( n \) for which \( f(n) \neq 0 \)), which in particular improved considerably the estimate (1). Such sums were further studied by De Koninck and Galambos [2].

In this paper, we prove the following:

Theorem. Let \( a \) be an arbitrary positive integer; then

\[
\sum_{n \leq x} \Omega(n)/\omega(n) = x + x \sum_{i=1}^{a} a_i/(\log \log x)^i + O(x/(\log \log x)^{a+1}),
\]

where \( a_1 = \sum \frac{1}{p(p-1)} \) and all the other \( a_i \)'s are computable constants.
PROOF. Let $t$ and $u$ be real numbers satisfying $|t| \leq 1$, $|u| \leq 1$. Then, for $\text{Re } s > 1$, we have

$$\sum_{n=1}^{\infty} \frac{t^{\omega(n)}}{n^s} = \prod_p \left( 1 + \frac{tu}{p^s} + \frac{t^2 u}{p^{2s}} + \frac{t^3 u}{p^{3s}} + \cdots \right)$$

$$= (\zeta(s))^{tu} \prod_p \left( 1 - \frac{1}{p^s} \right)^{tu} \prod_p \left( 1 + \frac{tu}{p^s} + \frac{t^2 u}{p^{2s}} + \frac{t^3 u}{p^{3s}} + \cdots \right)$$

$$= (\zeta(s))^{tu} H(t, u; s),$$

say (Here $\zeta(s)$ denotes the Riemann zeta-function.)

Using a theorem of A. Selberg [5], as we did previously in [1], we obtain that

$$\sum_{n \geq x} t^{\omega(n)} n^{\omega(n)} = (H(t, u; 1)/\Gamma(tu)) x \log^{tu-1} x + O(x \log^{tu-2} x),$$

uniformly for $|t| \leq 1$, $|u| \leq 1$, which certainly implies that

$$\sum_{n \geq x} t^{\omega(n)} n^{\omega(n)} = \frac{H(t, u; 1)}{\Gamma(tu)} x \log^{tu-1} x + O(x/\log x)$$

(2)

uniformly for $|t| \leq 1$, $|u| \leq 1$.

Now differentiating both sides of (2) with respect to $t$ gives

$$\sum_{n \geq x} \Omega(n) t^{\omega(n)-1} n^{\omega(n)} = \frac{x}{\log x} \left\{ \log^{tu} x \frac{d}{dt} \left( \frac{H(t, u; 1)}{\Gamma(tu)} \right) \right\}$$

$$+ \frac{H(t, u; 1)}{\Gamma(tu)} \cdot \log^{tu} x \cdot \log \log x \cdot u + O(1),$$

which, by setting $t=1$ and dividing both sides by $u$, becomes

$$\sum_{n \geq x} \Omega(n) u^{\omega(n)-1} n^{\omega(n)}$$

(3)

$$= (x/\log x) \{ G(u) \log^u x + F(u) \log^u x \cdot \log \log x + O(1/u) \}$$

uniformly for $|u| \leq 1$, where

$$G(u) = \left. \frac{1}{u} \frac{d}{dt} \left( \frac{H(t, u; 1)}{\Gamma(tu)} \right) \right|_{t=1}$$

and

$$F(u) = \frac{H(1, u; 1)}{\Gamma(u)}$$
We now proceed to integrate both sides of (3) with respect to \( u \) between 
\( \varepsilon(x) = (\log x)^{-1/2} \) and 1 (\( x \geq 3 \)). First we have

\[
\sum_{n \geq 2} \frac{\Omega(n)}{\omega(n)} \int_{\varepsilon(x)}^{1} u^{\omega(n)-1} du = \sum_{n \geq 2} \frac{\Omega(n)}{\omega(n)} - \sum_{n \geq 2} \frac{\Omega(n)}{\omega(n)} (\varepsilon(x)^{\omega(n)}
\]

since \( \omega(n) \geq 1 \) for \( n \geq 2 \). It can be proved [4] in an elementary way that 
\( \sum_{2 \leq n \leq x} \Omega(n) = O(x \log \log x) \). Therefore,

\[
\int_{\varepsilon(x)}^{1} \left( \sum_{n \geq 2} \frac{\Omega(n)}{\omega(n)} u^{\omega(n)-1} \right) du = \sum_{n \geq 2} \frac{\Omega(n)}{\omega(n)} + O(x(\log \log x)(\log x)^{-1/2})
\]

(4)

On the other hand, as in [1], repeated integration by parts yields

\[
\int_{\varepsilon(x)}^{1} G(u) \log^{a} x du = \log x \left( \frac{G(1)}{\log \log x} - \frac{G'(1)}{(\log \log x)^{2}} + \frac{G''(1)}{(\log \log x)^{3}} - \cdots \right)
\]

+ \( \frac{(-1)^{a-1}G^{(a-1)}(1)}{(\log \log x)^{a}} + O\left( \frac{1}{(\log \log x)^{a+1}} \right) \)

(5)

Similarly we obtain

\[
\log \log x \int_{\varepsilon(x)}^{1} F(u) \log^{a} x du
\]

(6) \[
= \log x \left( F(1) - \frac{F'(1)}{\log \log x} + \frac{F'(1)}{(\log \log x)^{2}} - \cdots \right)
\]

+ \( \frac{(-1)^{a}F^{(a)}(1)}{(\log \log x)^{a}} + O\left( \frac{1}{(\log \log x)^{a+1}} \right) \).
Finally,
\[
\frac{x}{\log x} \int_{x}^{1} \frac{du}{\log u} = O\left(\frac{x \log \varepsilon(x)}{\log x}\right) = O\left(\frac{x \log \log x}{\log x}\right) 
\]
(7)
\[
= O\left(\frac{1}{(\log \log x)^{2+1}}\right)
\]
Putting together relations (3), (4), (5), (6) and (7), we have that
\[
\sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} = x \left( F(1) + \frac{G(1) - F'(1)}{\log \log x} - \frac{G'(1) - F''(1)}{(\log \log x)^2} + \cdots 
\right.
\]
\[
+ (-1)^{\sigma-1} \frac{G^{(\sigma-1)}(1) - F^{(\sigma)}(1)}{(\log \log x)^\sigma} + O\left(\frac{1}{(\log \log x)^{2+1}}\right) \right)
\]
A quite simple computation shows that \( F(1) = 1 \) and that \( G(1) - F'(1) = \sum_p 1/p(p-1) \), which proves our Theorem.

From the above reasoning it is clear that similar estimates of \( \sum_{n \leq x} g(n)f(n) \) could be obtained for a larger class of additive functions \( f \) and \( g \) along the lines of our previous paper [1].

REFERENCES


Département de Mathématiques, Université Laval, G1K 7P4, Québec, Canada