SUMS OF QUOTIENTS OF ADDITIVE FUNCTIONS

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Abstract. Denote by \( \omega(n) \) and \( \Omega(n) \) the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \), respectively. Given any positive integer \( \alpha \), we prove that

\[
\sum_{2 \leq n \leq x} \frac{\Omega(n)}{\omega(n)} = x + x \sum_{i=1}^{\alpha} a_i (\log \log x)^i + O(x/(\log \log x)^{\alpha+1}),
\]

where \( a_1 = \sum_p 1/p(p-1) \) and all the other \( a_i \)'s are computable constants. This improves a previous result of R. L. Duncan.

Denote by \( \omega(n) \) and \( \Omega(n) \) the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \), respectively. R. L. Duncan [3] proved that

\[
\sum_{2 \leq n \leq x} \frac{\Omega(n)}{\omega(n)} = x + O(x/\log \log x).
\]

Duncan's result was based on the elementary estimate

\[
\sum_{2 \leq n \leq x} 1/\omega(n) = O(x/\log \log x).
\]

In a previous paper [1], we gave estimates of \( \sum_{n \leq x} 1/f(n) \) for a large class of additive functions \( f(n) \) (where \( \sum' \) denotes summation over those values of \( n \) for which \( f(n) \neq 0 \)), which in particular improved considerably the estimate (1). Such sums were further studied by De Koninck and Galambos [2].

In this paper, we prove the following:

Theorem. Let \( \alpha \) be an arbitrary positive integer; then

\[
\sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} = x + x \sum_{i=1}^{\alpha} a_i (\log \log x)^i + O(x/(\log \log x)^{\alpha+1}),
\]

where \( a_1 = \sum_p 1/p(p-1) \) and all the other \( a_i \)'s are computable constants.
PROOF. Let \( t \) and \( u \) be real numbers satisfying \( |t| \leq 1, |u| \leq 1 \). Then, for \( \text{Re } s > 1 \), we have

\[
\sum_{n=1}^{\infty} \frac{\prod_{n}^{\infty} \left( 1 + \frac{tu}{p^n} + \frac{t^2u}{p^{2n}} + \frac{t^3u}{p^{3n}} + \cdots \right)}{n^s} = (\zeta(s))^u \prod_{p} \left( 1 - \frac{1}{p^s} \right) \prod_{p} \left( 1 + \frac{tu}{p^s} + \frac{t^2u}{p^{2s}} + \frac{t^3u}{p^{3s}} + \cdots \right)
\]

\[
= (\zeta(s))^u H(t, u; s),
\]

say (Here \( \zeta(s) \) denotes the Riemann zeta-function.)

Using a theorem of A. Selberg [5], as we did previously in [1], we obtain that

\[
\sum_{n \leq x} \prod_{n}^{\infty} \left( 1 + \frac{tu}{p^n} + \frac{t^2u}{p^{2n}} + \frac{t^3u}{p^{3n}} + \cdots \right) = \frac{H(t, u; 1)}{\Gamma(tu)} x \log^{tu-1} x + O(x \log^{tu-2} x),
\]

uniformly for \( |t| \leq 1, |u| \leq 1 \), which certainly implies that

\[
\sum_{n \leq x} \prod_{n}^{\infty} \left( 1 + \frac{tu}{p^n} + \frac{t^2u}{p^{2n}} + \frac{t^3u}{p^{3n}} + \cdots \right) = \frac{H(t, u; 1)}{\Gamma(tu)} x \log^{tu} x + O(1),
\]

uniformly for \( |t| \leq 1, |u| \leq 1 \).

Now differentiating both sides of (2) with respect to \( t \) gives

\[
\sum_{n \leq x} \Omega(n) \prod_{n}^{\infty} \left( 1 + \frac{tu}{p^n} + \frac{t^2u}{p^{2n}} + \frac{t^3u}{p^{3n}} + \cdots \right) = \frac{x}{\log x} \left( \log^{tu} x \frac{d}{dt} \left( \frac{H(t, u; 1)}{\Gamma(tu)} \right) \right)
\]

\[
+ \frac{H(t, u; 1)}{\Gamma(tu)} \cdot \log^{tu} x \cdot \log \log x \cdot u + O(1),
\]

which, by setting \( t = 1 \) and dividing both sides by \( u \), becomes

\[
\sum_{n \leq x} \Omega(n) \prod_{n}^{\infty} \left( 1 + \frac{tu}{p^n} + \frac{t^2u}{p^{2n}} + \frac{t^3u}{p^{3n}} + \cdots \right) = (x/\log x) \{ G(u) \log^{u} x + F(u) \log^{u} x \cdot \log \log x + O(1/u) \}
\]

uniformly for \( |u| \leq 1 \), where

\[
G(u) = \frac{1}{u} \left. \frac{d}{dt} \left( \frac{H(t, u; 1)}{\Gamma(tu)} \right) \right|_{t=1}
\]

and

\[
F(u) = \frac{H(1, u; 1)}{\Gamma(u)}
\]
We now proceed to integrate both sides of (3) with respect to $u$ between $e(x) = (\log x)^{1/2}$ and 1 ($x \geq 3$). First we have

\[
\int_{e(x)}^{1} \left( \sum_{2 \leq n \leq x} \Omega(n) u^{\omega(n)-1} \right) du = \sum_{2 \leq n \leq x} \Omega(n) \int_{e(x)}^{1} u^{\omega(n)-1} du
\]

\[
= \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} - \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} (e(x))^{\omega(n)}
\]

\[
= \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} + O\left( (e(x) \sum_{2 \leq n \leq x} \Omega(n)) \right),
\]

since $\omega(n) \geq 1$ for $n \geq 2$. It can be proved [4] in an elementary way that $\sum_{2 \leq n \leq x} \Omega(n) = O(x \log \log x)$. Therefore,

\[
\int_{e(x)}^{1} \left( \sum_{2 \leq n \leq x} \Omega(n) u^{\omega(n)-1} \right) du = \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} + O(x(\log \log x)(\log x)^{-1/2})
\]

(4)

\[
= \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} + O\left( \frac{x}{(\log \log x)^{x+1}} \right).
\]

On the other hand, as in [1], repeated integration by parts yields

\[
\int_{e(x)}^{1} G(u) \log^{u} x du
\]

\[
= \log x \left( \frac{G(1)}{\log \log x} - \frac{G'(1)}{(\log \log x)^{2}} + \frac{G''(1)}{(\log \log x)^{3}} - \cdots \right.
\]

\[
\left. + \frac{(-1)^{x-1} G((x-1)(1))}{(\log \log x)^{x}} + O\left( \frac{1}{(\log \log x)^{x+1}} \right) \right)
\]

(5)

\[
\left. + \frac{(-1)^{x+1}}{(\log \log x)^{x+1}} \int_{e(x)}^{1} G^{(x+1)}(u) \log^{u} x du \right)
\]

\[
= \log x \left( \frac{G(1)}{\log \log x} - \frac{G'(1)}{(\log \log x)^{2}} + \cdots \right.
\]

\[
\left. + \frac{(-1)^{x-1} G((x-1)(1))}{(\log \log x)^{x}} + O\left( \frac{1}{(\log \log x)^{x+1}} \right) \right).
\]

Similarly we obtain

\[
\log \log x \int_{e(x)}^{1} F(u) \log^{u} x du
\]

(6)

\[
= \log x \left( F(1) - \frac{F'(1)}{\log \log x} + \frac{F'(1)}{(\log \log x)^{2}} - \cdots \right.
\]

\[
\left. + \frac{(-1)^{x} F((x)(1))}{(\log \log x)^{x}} + O\left( \frac{1}{(\log \log x)^{x+1}} \right) \right).
\]
Finally,
\[
\frac{x}{\log x} \int_{\epsilon(x)}^{1} \frac{du}{u} = O\left(\frac{x \log \epsilon(x)}{\log x}\right) = O\left(\frac{x \log \log x}{\log x}\right)
\]
(7)
\[
= O\left(\frac{1}{(\log \log x)^{a+1}}\right)
\]

Putting together relations (3), (4), (5), (6) and (7), we have that
\[
\sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} = x\left(F(1) + \frac{G(1) - F'(1)}{\log \log x} - \frac{G'(1) - F''(1)}{(\log \log x)^2} + \cdots + (-1)^{a-1} \frac{G^{(a-1)}(1) - F^{(a)}(1)}{(\log \log x)^a} + O\left(\frac{1}{(\log \log x)^{a+1}}\right)\right)
\]

A quite simple computation shows that \(F(1)=1\) and that \(G(1) - F'(1) = \sum_p 1/p(p-1)\), which proves our Theorem.

From the above reasoning it is clear that similar estimates of \(\sum_{n \leq x} g(n)f(n)\) could be obtained for a larger class of additive functions \(f\) and \(g\) along the lines of our previous paper [1].

REFERENCES


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