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NONSPLITTING SEQUENCES OF VALUE GROUPS

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ABSTRACT. If K is the quotient field of an integral domain D, then the value group $V_K(D)$ of D in K is the group $K^*/U(D)$, partially ordered by $D^*/U(D)$, where $U(D)$ denotes the group of units of D. This note shows that if the sequence

$$(1) \qquad \{1\} \to G \to H \to J \to \{1\}$$

is lexicographically exact and if H is lattice ordered, then there is a Bezout domain B and a prime ideal P of B such that $V_K(B)=H$, $V_K(B_P)=J$, and $V_k(B/P)=G$, where k denotes the residue field of B_P. Moreover, B is the direct sum of B/P and P, and $B_P=k+P$. In particular, the sequence (1) need not split, even with somewhat stringent restrictions on the integral domain B. This gives a negative answer to a question posed by R. Gilmer.

If K is a field containing a subring D with identity, the value group of D in K is the group $K^*/U(D)$, partially ordered by $D^*/U(D)$. We denote this group by $V_K(D)$ and remark that if K is the quotient field of D, $V_K(D)$ has traditionally been called the group of divisibility of D. The set of all principal D-submodules of K, partially ordered by the set of all principal (integral) ideals of D, is order isomorphic to $V_K(D)$.

Suppose that C is a local domain with maximal ideal M and quotient field K. Let $k=C/M$ and let A be an integral domain with quotient field k. If σ denotes the natural homomorphism from C onto k, and if $B=\sigma^{-1}(A)$, then we say that B is the *composite of C and A over M*. If, moreover, B is the direct sum of A and M, then we say that B is the *direct composite of C and A over M*. In general, if B is a subring of C, we have the exact sequence of abelian groups:

$$(1) \qquad \{1\} \longrightarrow U(C)/U(B) \longrightarrow K^*/U(B) \xrightarrow{\beta} K^*/U(C) \longrightarrow \{1\},$$

where $\beta(xU(B))=xU(C)$. But if B is the composite of C and A over the maximal M of C, then the group $U(C)/U(B)$, partially ordered by

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$[U(C)\cap B]/U(B)$, is order isomorphic to $V_k(A)$, and, as Ohm observes in [**5**, p. 581], the sequence:

$$\{1\} \longrightarrow V_k(A) \longrightarrow V_K(B) \xrightarrow{\beta} V_K(C) \longrightarrow \{1\}, \tag{2}$$

is lexicographically exact.

In this same paper, Ohm [**5**, p. 582] gives necessary and sufficient conditions for (2) to split. He then uses this result to prove the following: If J is a totally ordered abelian group, and if G is the value group of some integral domain A with quotient field k, then there is an integral domain B with value group $G \oplus_L J$, the lexicographic sum of G and J.

In the proof, Ohm uses a famous theorem of Krull [**2**, p. 164] to construct a valuation domain C so that $V_K(C)=J$ and $C/M=k$, where M is the maximal ideal of C. The domain B, then, is the composite of C and A over M. By further analyzing the proof we see that B is, in fact, the direct composite of C and A over M.

The following question arises naturally: *If B is the direct composite of C and A over the maximal ideal M of C, does the sequence* (2) *necessarily split*? Originally, R. Gilmer posed this question while working on a paper with Bastida [**1**].

This question has a negative answer, but before we show this, let us interpret Ohm's result somewhat more broadly.

Suppose that

$$\{1\} \longrightarrow G \longrightarrow H \xrightarrow{\beta} J \longrightarrow \{1\} \tag{3}$$

is a lexicographically exact sequence of partially ordered abelian groups. Ohm's result shows, in essence, that *if* (3) *splits*, then, under certain conditions, (3) is a sequence of value groups: $G=V_k(A)$, $H=V_K(B)$, $J=V_K(C)$, where B is the (direct) composite of C and A over M.

Now we ask: *Is there an analogous result where* (3) *does not split*? We answer this question affirmatively when H is lattice ordered; then we use this result to answer Gilmer's question in the negative.

The essential clue to the argument is the Krull-Kaplansky-Jaffard-Ohm theorem. (See [**3**, p. 197] for the history of the development of this theorem.) This result asserts that for any lattice ordered abelian group H and for any field F, the map v from the group algebra $F[X; H]$ onto H defined by $v(\sum_i a_i X^{h_i})=\inf\{h_i\}$ can be extended to a semivaluation on the quotient field K of $F[X; H]$ such that the integral domain $B=\{y \in K \mid v(y) \geqq 1\} \cup \{0\}$ has value group H. Actually, the domain B, so constructed, is a Bezout domain.

Now let us list some additional results that will be useful in our argument.

(1) If P is a prime ideal of an integral domain B, then B is the composite of B_P and B/P if and only if P compares with each ideal of B.

Recall that if B has quotient field K, then B is a GCD-domain if and only if $V_K(B)$ is lattice ordered. By definition, B is a Bezout domain if and only if each finitely generated ideal of B is principal, but the following equivalent form is more useful in the present context.

(2) Suppose that B is a GCD-domain with quotient K, and suppose, moreover, that v is the associated semivaluation from K^* onto $V_K(B)$. Then, B is a Bezout domain if and only if for each nonzero ideal Q of B, $V(Q\backslash\{0\})$ is a filter in the positive cone $V(B^*)$ in $V_K(B)$.

(3) If the sequence (3) is lexicographically exact, where H is lattice ordered, then J is totally ordered and G is a l-ideal of H. Moreover, each filter of H_+ compares, under containment, with the prime filter $H_+\backslash G_+$.

Now we are prepared to answer the second question. Suppose that the sequence (3) is lexicographically exact and that H is lattice ordered. If F is any field, let K denote the quotient field of the group algebra $F[X; H]$. Use the Krull-Kaplansky-Jaffard-Ohm theorem to construct a domain B such that $V_K(B)=H$. Let P be the prime ideal of B such that $v(P\backslash\{0\})=H_+\backslash G_+$, where v denotes the canonical semivaluation from K^* onto $V_K(B)=H$.

By [4], J is the value group of $C=B_P$, and, since J is totally ordered, C is a valuation domain.

Since every filter in H_+ compares with $H_+\backslash G_+$, it follows that every ideal of B compares with P. Thus, B is the composite of $C=B_P$ and $A=B/P$ over P. Moreover, P is the maximal ideal of C.

Next observe that B is the direct composite of C and A over P. Let k denote the quotient field of $F[X; G]$, and show that C is the direct sum of k and P. To do this, let $w=\beta\cdot v$, and observe that C is the valuation domain associated with w.

If $g_1\in G$, then $w(X^{g_1})=\beta(g_1)=1$, so that $k^*\subseteq U(C)$, $k\cap P=(0)$, and $k+P\subseteq C$. If $f/g\in C$, where f and g are elements of $F[X; H]$, then $w(f/g)\geqq 1$. If $w(f/g)>1$, then $f/g\in P$, and $f/g\in k+P$. But if $w(f)w(g)^{-1}=1$, then $v(f)v(g)^{-1}\in G$. Furthermore, if $v(f)=h_1$ and $v(g)=h_2$, then, by considering f/X^{h_2} and g/X^{h_2}, we see that without loss of generality we may assume that $w(f)=w(g)=1$. Next, observe that f and g are in $k+P$. Write $f=f_1+f_2$, where $f_2\in F[X; G]$ and $f_1=\sum_i a_iX^{h_i}$, where no h_i is in G. Then $w(f_1)>1$, since $v(f_1)=\inf\{h_i\}\notin G$, a prime subgroup of H. Therefore, $f_1\in P$ and $f\in k+P$. Clearly, $g\in k+P$ and $w(g)=1$ imply that $1/g\in k+P$. Thus, $B=A+P$ and $G=V_k(A)$.

In sum: If (3) is lexicographically exact, and if H is lattice ordered, there is a Bezout domain B such that: (i) $V_K(B)=H$, (ii) B is the direct composite of $C=B_P$ and $A=B/P$, where P is the prime ideal of B associated with the prime subgroup G of H, and (iii) $V_K(C)=J$ and $V_k(A)=G$.

In particular, suppose that $\text{Ext}(J, G)\neq 0$, where G and J are torsion free abelian groups, and suppose, moreover, that H is a nonsplit extension of G by J. Then, let each of G and J be totally ordered, and let H be totally ordered by the set

$$H_+ = \{x \in H \mid x \in G_+ \text{ or } \beta(x) \in J_+\backslash\{1\}\}.$$

The sequence (3), therefore, is lexicographically exact and the rings A, B, and C, as constructed above, answer Gilmer's question in the negative.

Finally, we ask: Given a lexicographically exact sequence $\{1\}\to G\to H\to J\to\{1\}$, where H is a value group, under what conditions do there exist domains A, B, C having value groups G, H, J respectively such that B is the *direct* composite of C and A? This paper shows that H being lattice if sufficient, while Ohm [**5**, §4] has shown that if one omits the word direct, then, for example, all that is needed is that G be filtered.

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