NONSPLITTING SEQUENCES OF VALUE GROUPS

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Abstract. If $K$ is the quotient field of an integral domain $D$, then the value group $V_K(D)$ of $D$ in $K$ is the group $K^* / U(D)$, partially ordered by $D^* / U(D)$, where $U(D)$ denotes the group of units of $D$. This note shows that if the sequence

$$
1 \rightarrow G \rightarrow H \rightarrow J \rightarrow 1
$$

is lexicographically exact and if $H$ is lattice ordered, then there is a Bezout domain $B$ and a prime ideal $P$ of $B$ such that $V_K(B) = H$, $V_K(B_P) = J$, and $V_K(B/P) = G$, where $k$ denotes the residue field of $B_P$. Moreover, $B$ is the direct sum of $B/P$ and $P$, and $B_P = k + P$. In particular, the sequence (1) need not split, even with somewhat stringent restrictions on the integral domain $B$. This gives a negative answer to a question posed by R. Gilmer.

If $K$ is a field containing a subring $D$ with identity, the value group of $D$ in $K$ is the group $K^* / U(D)$, partially ordered by $D^* / U(D)$. We denote this group by $V_K(D)$ and remark that if $K$ is the quotient field of $D$, $V_K(D)$ has traditionally been called the group of divisibility of $D$. The set of all principal $D$-submodules of $K$, partially ordered by the set of all principal (integral) ideals of $D$, is order isomorphic to $V_K(D)$.

Suppose that $C$ is a local domain with maximal ideal $M$ and quotient field $K$. Let $k = C/M$ and let $A$ be an integral domain with quotient field $k$. If $\sigma$ denotes the natural homomorphism from $C$ onto $k$, and if $B = \sigma^{-1}(A)$, then we say that $B$ is the composite of $C$ onto $k$. If, moreover, $B$ is the direct sum of $A$ and $M$, then we say that $B$ is the direct composite of $C$ and $A$ over $M$. In general, if $B$ is a subring of $C$, we have the exact sequence of abelian groups:

$$
\{1\} \rightarrow U(C) / U(B) \rightarrow K^* / U(B) \xrightarrow{\beta} K^* / U(C) \rightarrow \{1\},
$$

where $\beta(xU(B)) = xU(C)$. But if $B$ is the composite of $C$ and $A$ over the maximal $M$ of $C$, then the group $U(C) / U(B)$, partially ordered by
[\U(C) \cap B]/U(B), is order isomorphic to \(V_k(A)\), and, as Ohm observes in [5, p. 581], the sequence:

\[
\{1\} \rightarrow V_k(A) \rightarrow V_K(B) \xrightarrow{\beta} V_K(C) \rightarrow \{1\},
\]

is lexicographically exact.

In this same paper, Ohm [5, p. 582] gives necessary and sufficient conditions for (2) to split. He then uses this result to prove the following: If \(J\) is a totally ordered abelian group, and if \(G\) is the value group of some integral domain \(A\) with quotient field \(k\), then there is an integral domain \(B\) with value group \(G \oplus_L J\), the lexicographic sum of \(G\) and \(J\).

In the proof, Ohm uses a famous theorem of Krull [2, p. 164] to construct a valuation domain \(C\) so that \(V_K(C)=J\) and \(C/M=k\), where \(M\) is the maximal ideal of \(C\). The domain \(B\), then, is the composite of \(C\) and \(A\) over \(M\). By further analyzing the proof we see that \(B\) is, in fact, the direct composite of \(C\) and \(A\) over \(M\).

The following question arises naturally: If \(B\) is the direct composite of \(C\) and \(A\) over the maximal ideal \(M\) of \(C\), does the sequence (2) necessarily split? Originally, R. Gilmer posed this question while working on a paper with Bastida [1].

This question has a negative answer, but before we show this, let us interpret Ohm's result somewhat more broadly.

Suppose that

\[
\{1\} \rightarrow G \rightarrow H \xrightarrow{\beta} J \rightarrow \{1\}
\]

is a lexicographically exact sequence of partially ordered abelian groups. Ohm's result shows, in essence, that if (3) splits, then, under certain conditions, (3) is a sequence of value groups: \(G=V_k(A)\), \(H=V_K(B)\), \(J=V_K(C)\), where \(B\) is the (direct) composite of \(C\) and \(A\) over \(M\).

Now we ask: Is there an analogous result where (3) does not split? We answer this question affirmatively when \(H\) is lattice ordered; then we use this result to answer Gilmer's question in the negative.

The essential clue to the argument is the Krull-Kaplansky-Jaffard-Ohm theorem. (See [3, p. 197] for the history of the development of this theorem.) This result asserts that for any lattice ordered abelian group \(H\) and for any field \(F\), the map \(v\) from the group algebra \(F[X; H]\) onto \(H\) defined by \(v(\sum a_i X^{h_i})=\inf\{h_i\}\) can be extended to a semivaluation on the quotient field \(K\) of \(F[X; H]\) such that the integral domain \(B=\{y \in K | v(y) \geq 1\} \cup \{0\}\) has value group \(H\). Actually, the domain \(B\), so constructed, is a Bezout domain.

Now let us list some additional results that will be useful in our argument.
(1) If $P$ is a prime ideal of an integral domain $B$, then $B$ is the composite of $B_P$ and $B/P$ if and only if $P$ compares with each ideal of $B$.

Recall that if $B$ has quotient field $K$, then $B$ is a GCD-domain if and only if $V_K(B)$ is lattice ordered. By definition, $B$ is a Bezout domain if and only if each finitely generated ideal of $B$ is principal, but the following equivalent form is more useful in the present context.

(2) Suppose that $B$ is a GCD-domain with quotient $K$, and suppose, moreover, that $v$ is the associated semivaluation from $K^*$ onto $V_K(B)$. Then, $B$ is a Bezout domain if and only if for each nonzero ideal $Q$ of $B$, $V(Q \setminus \{0\})$ is a filter in the positive cone $V(B^*)$ in $V_K(B)$.

(3) If the sequence (3) is lexicographically exact, where $H$ is lattice ordered, then $J$ is totally ordered and $G$ is a $\lambda$-ideal of $H$. Moreover, each filter of $H_+ \setminus G_+$ compares, under containment, with the prime filter $H_+ \setminus G_+$.

Now we are prepared to answer the second question. Suppose that the sequence (3) is lexicographically exact and that $H$ is lattice ordered. If $F$ is any field, let $K$ denote the quotient field of the group algebra $F[X; H]$. Use the Krull-Kaplansky-Jaffard-Ohm theorem to construct a domain $B$ such that $V_K(B) = H$. Let $P$ be the prime ideal of $B$ such that $v(P\{0\}) = H_+ \setminus G_+$, where $v$ denotes the canonical semivaluation from $A^*$ onto $V_K(B) = H$.

By [4], $J$ is the value group of $C = B_P$, and, since $J$ is totally ordered, $C$ is a valuation domain.

Since every filter in $H_+ \setminus G_+$ compares with $H_+ \setminus G_+$, it follows that every ideal of $B$ compares with $P$. Thus, $B$ is the composite of $C = B_P$ and $A = B/P$ over $P$. Moreover, $P$ is the maximal ideal of $C$.

Next observe that $B$ is the direct composite of $C$ and $A$ over $P$. Let $k$ denote the quotient field of $F[X; G]$, and show that $C$ is the direct sum of $k$ and $P$. To do this, let $w = \beta \cdot v$, and observe that $C$ is the valuation domain associated with $w$.

If $g_1 \in G$, then $w(X^{g_1}) = \beta(g_1) = 1$, so that $k^* \subseteq U(C)$, $k \cap P = \langle 0 \rangle$, and $k + P \subseteq C$. If $f \mid g \in C$, where $f$ and $g$ are elements of $F[X; H]$, then $w(f/g) \geq 1$. If $w(f/g) > 1$, then $f/g \in P$, and $f/g \in k + P$. But if $w(f)w(g)^{-1} = 1$, then $v(f) = v(g)^{-1}$. Furthermore, if $v(f) = h_1$ and $v(g) = h_2$, then, by considering $f/X^{h_2}$ and $g/X^{h_2}$, we see that without loss of generality we may assume that $w(f) = w(g) = 1$. Next, observe that $f$ and $g$ are in $k + P$. Write $f = f_1 + f_2$, where $f_2 \in F[X; G]$ and $f_1 = \sum a_i X^{h_i}$, where no $h_i$ is in $G$. Then $w(f_1) > 1$, since $v(f_1) = \inf\{h_i\} \notin G$, a prime subgroup of $H$. Therefore, $f_1 \in P$ and $f_2 \in k + P$. Clearly, $g \in k + P$ and $w(g) = 1$ imply that $1/g \in k + P$. Thus, $B = A + P$ and $G = V_k(A)$.

In sum: If (3) is lexicographically exact, and if $H$ is lattice ordered, there is a Bezout domain $B$ such that: (i) $V_K(B) = H$, (ii) $B$ is the direct composite
of $C = B_P$ and $A = B/P$, where $P$ is the prime ideal of $B$ associated with the prime subgroup $G$ of $H$, and (iii) $V_K(C) = J$ and $V_k(A) = G$.

In particular, suppose that $\text{Ext}(J, G) \neq 0$, where $G$ and $J$ are torsion free abelian groups, and suppose, moreover, that $H$ is a nonsplit extension of $G$ by $J$. Then, let each of $G$ and $J$ be totally ordered, and let $H$ be totally ordered by the set

$$H_+ = \{ x \in H \mid x \in G_+ \text{ or } \beta(x) \in J_+ \setminus \{1\} \}.$$ 

The sequence (3), therefore, is lexicographically exact and the rings $A$, $B$, and $C$, as constructed above, answer Gilmer’s question in the negative.

Finally, we ask: Given a lexicographically exact sequence $\{1\} \rightarrow G \rightarrow H \rightarrow J \rightarrow \{1\}$, where $H$ is a value group, under what conditions do there exist domains $A$, $B$, $C$ having value groups $G$, $H$, $J$ respectively such that $B$ is the direct composite of $C$ and $A$? This paper shows that $H$ being lattice if sufficient, while Ohm [5, §4] has shown that if one omits the word direct, then, for example, all that is needed is that $G$ be filtered.

REFERENCES


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