SOME IMPLICATIONS OF THE EULER-POINCARÉ CHARACTERISTIC FOR COMPLETE INTERSECTION MANIFOLDS

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Abstract. Studies on relations between Euler-Poincaré characteristic and codimension of a complete intersection manifold in a complex projective space.

1. Introduction. Let \( P_{n+p}(C) \) be an \((n+p)\)-dimensional complex projective space. An \( n \)-dimensional algebraic manifold \( M \) imbedded in \( P_{n+p}(C) \) is called a complete intersection manifold if \( M \) is given as an intersection of \( p \) nonsingular hypersurfaces \( M_1, \cdots, M_p \) in general position in

\[
P_{n+p}(C) : M = M_1 \cap \cdots \cap M_p.
\]

It is known that the Chern classes of a complete intersection manifold \( M \) are completely determined by the degrees of \( M_1, \cdots, M_p \). In particular, the Euler-Poincaré characteristic of a complete intersection manifold is completely determined by the degrees of \( M_1, \cdots, M_p \).

In §2 we prove a formula for the Euler-Poincaré characteristic of a complete intersection manifold in terms of the degrees.

It is sometimes very important to know the smallest codimension for a complete intersection manifold: the smallest \( p \) for which \( M \) can be imbedded as a complete intersection manifold in \( P_{n+p}(C) \). In §3 we prove several results in this direction.

2. The Euler-Poincaré characteristic of complete intersection manifolds. Let \( P_{n+p}(C) \) be an \((n+p)\)-dimensional complex projective space and let \( M \) be an \( n \)-dimensional complete intersection manifold imbedded in \( P_{n+p}(C) : M = M_1 \cap \cdots \cap M_p \), where the \( M_\alpha \)'s are nonsingular hypersurfaces in \( P_{n+p}(C) \). The following theorem gives a concrete formula for the Euler-Poincaré characteristic of a complete intersection manifold.
THEOREM 2.1.³ Let $M = M_1 \cap \cdots \cap M_p$ be an $n$-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. If $\text{deg } M_a = a_1$, then the Euler-Poincaré characteristic $\chi(M)$ of $M$ is given by

$$\chi(M) = \left[ \sum_{k=0}^{n} (-1)^k \binom{n+p+1}{n-k} a_k \right] \left( \prod_{a=1}^{p} a_a \right),$$

where $a_k = \sum_{a_1 \leq \cdots \leq a_k} a_{a_1} a_{a_2} \cdots a_{a_k}$ (the sum of all homogeneous monomials of degree $k$ in $a_1, a_2, \cdots, a_p$) and $\binom{n+p+1}{n-k}$ is the binomial coefficient.

PROOF. Let $\tilde{h}$ be the generator of $H^2(P_{n+p}(C), Z)$ corresponding to the divisor class of a hyperplane $P_{n+p-1}(C)$. Then the total Chern class $c(P_{n+p}(C))$ of $P_{n+p}(C)$ is given by

$$c(P_{n+p}(C)) = (1 + \tilde{h})^{n+p+1}$$

Let $j: M \rightarrow P_{n+p}(C)$ be the imbedding and $v$ be the normal bundle of $j(M)$ in $P_{n+p}(C)$. Then the total Chern class $c(v)$ of $v$ is given by

$$c(v) = (1 + a_1 h) \cdots (1 + a_p h),$$

where $h$ is the image of $\tilde{h}$ under the homomorphism $j^*: H^2(P_{n+p}(C), Z) \rightarrow H^2(M, Z)$. Since $j^* T(P_{n+p}(C)) = T(M) \oplus v$ (Whitney sum), we have

$$j^* c(P_{n+p}(C)) = c(M) \cdot c(v),$$

where $c(M)$ is the total Chern class of $M$. Let $c_i(M)$ be the $i$th Chern class of $M$. Then we have

$$(1 + h)^{n+p+1} = [1 + c_1(M) + \cdots + c_n(M)] \cdot (1 + a_1 h) \cdots (1 + a_p h)$$

which implies that

$$c_n(M) = \left[ \sum_{k=0}^{n} (-1)^k \binom{n+p+1}{n-k} a_k \right] h^n.$$

Taking the values of both sides on the fundamental cycle of $M$, we have

$$\chi(M) = \left[ \sum_{k=0}^{n} (-1)^k \binom{n+p+1}{n-k} a_k \right] \left( \prod_{a=1}^{p} a_a \right).$$

REMARK. Let $b_i(M)$ be the $i$th Betti number of $M$. It is known that if $M$ is an $n$-dimensional complete intersection manifold, then

$$b_{2k}(M) = 1 \quad (2k \neq n),$$

$$b_{2k+1}(M) = 0 \quad (2k + 1 \neq n).$$

³ Although Theorem 2.1 can be obtained from Theorem 22.1.1 of [1], we give here a direct proof for the sake of completeness.
Therefore \( \chi(M) = \sum_{i=0}^{2n} (-1)^i b_i(M) \geq n+1 \) (resp. \( \leq n+1 \)) provided that \( n \) is even (resp. odd).

3. Some implications of the Euler-Poincaré characteristic. First we prove the following:

**Theorem 3.1.** Let \( M \) be an \( n \)-dimensional complete intersection manifold. If \( \chi(M) = v_1 \cdots v_p \) for some prime numbers \( v_1, \ldots, v_p \) \((\neq \pm 1)\), then \( M \) can be imbedded as a complete intersection manifold in \( P_{n+p}(C) \).

**Proof.** We can assume without loss of generality that \( M \) is a complete intersection manifold imbedded in \( P_{n+q}(C) \) for some \( q \geq p \). In fact, if \( M \) is a complete intersection manifold imbedded in \( P_{n+r}(C) \) for some \( r < p \), then, by imbedding \( P_{n+r}(C) \) into \( P_{n+q}(C) \) as a linear subspace for some \( q \geq p \), \( M \) can be considered as a complete intersection manifold imbedded in \( P_{n+q}(C) \).

Theorem 2.1 implies that \( \chi(M) \) is given as a product of \( q+1 \) integers:
\[
\chi(M) = [-\cdots]a_1 \cdots a_q.
\]
On the other hand, since \( \chi(M) = v_1 \cdots v_p \), at least \( q-p \) \( a_x \)'s must be equal to 1. This implies that \( M \) can be imbedded as a complete intersection manifold in some \((n+p)\)-dimensional linear subspace \( P_{n+p}(C) \) in \( P_{n+q}(C) \). Q.E.D.

It is sometimes very important to know the smallest codimension for which \( M \) can be imbedded as a complete intersection manifold. From this point of view, Theorem 3.1 may not be the best possible in general. We shall prove several partial improvements of Theorem 3.1 in the following:

First we prove the following:

**Lemma 3.2.** Let \( M \) be an \( n \)-dimensional complete intersection manifold imbedded in \( P_{n+p}(C) \). Assume that \( \chi(M) = v_1 \cdots v_p \) for some prime numbers \( v_1, \ldots, v_p \) \((\neq \pm 1)\). If the Diophantine equation \( \sum_{k=0}^{n} (-1)^k (\binom{n+p+1}{n-k}) \sigma_k = \pm 1 \) has no solution satisfying \( a_1 \geq 2, \cdots, a_p \geq 2 \), then \( M \) can be imbedded as a complete intersection manifold in \( P_{n+p-1}(C) \).

**Proof.** Theorem 2.1 implies that \( \chi(M) \) is given as a product of \( p+1 \) integers: \( \chi(M) = [-\cdots]a_1 \cdots a_p \). On the other hand, since \( \chi(M) = v_1 \cdots v_p \), at least one among \( [-\cdots] \), \( a_1, \ldots, a_p \) must be equal to \( \pm 1 \). If the Diophantine equation \( [-\cdots] = \pm 1 \) has no solution satisfying \( a_1 > 1, \cdots, a_p > 1 \), then at least one \( a_x \) must be equal to 1. Q.E.D.

**Theorem 3.3.** Let \( M \) be a complete intersection manifold. If \( \chi(M) \) is a prime number, then \( M \) is one of the following:

1. a linear subspace
2. a quadric in \( P_2(C) \),
3. a curve of degree 4 in \( P_2(C) \).
Proof. In consideration of Theorem 3.1, we may assume that $M$ is a nonsingular hypersurface of $P_{n+1}(C)$. Putting $a=a_1$, from Theorem 2.1 we have

$$\chi(M) = \left[\sum_{k=0}^{n} (-1)^{k} \binom{n+2}{n-k} a_k \right] a = \frac{(1-a)^{n+2} - 1 + (n+2)a}{a^2} \cdot a.$$ 

Since $\chi(M)$ is a prime number, either $a=1$ or

$$\frac{(1-a)^{n+2} - 1 + (n+2)a}{a^2} = \pm 1.$$ 

If $a=1$, then $M$ is a linear subspace. Therefore we consider the latter case. It suffices to prove the following:

**Lemma.** The only positive integral solutions for the Diophantine equation $(1-a)^{n+2} - 1 + (n+2)a = \pm a^2$ are $(n, a)=(1, 2)$ and $(n, a)=(1, 4)$.

**Proof of Lemma.** Since $a=1$ cannot be a solution for the equation $(1-a)^{n+2} - 1 + (n+2)a = \pm a^2$, we consider this equation for $a \geq 2$. Let

$$f_n(a) = \frac{(1-a)^{n+2} - 1 + (n+2)a}{a^2}$$ for $a \geq 2$.

(i) If $n$ is even: It is easy to show that $f_n(a)$ is monotonically increasing and $f_n(2)=(n+2)/2>1$. Therefore the equation $(1-a)^{n+2} - 1 + (n+2)a = \pm a^2$ has no solution of the form $(n, a)=(\text{even}, *)$.

(ii) If $n$ is odd: $f_n(a)$ is monotonically decreasing and $f_n(2)=(n+1)/2$. It is also easy to show that $f_n(5) \leq -2$. Therefore the only candidates for the solutions for the equation $(1-a)^{n+2} - 1 + (n+2)a = \pm a^2$ are of the form $(n, a)=(\text{odd}, 2)$, $(n, a)=(\text{odd}, 3)$ or $(n, a)=(\text{odd}, 4)$. We can easily prove that the only solutions are $(n, a)=(1, 2)$ and $(n, a)=(1, 4)$. Q.E.D.

**Corollary 3.4.** Let $M$ be a complete intersection manifold. If $\dim M > 1$ and if $\chi(M)$ is a prime number, then $M$ is a linear subspace.

**Theorem 3.5.** Let $M$ be a complete intersection manifold. If $\dim M > 1$ and if $\chi(M) = \nu_1 \nu_2$ for some prime numbers $\nu_1, \nu_2$ ($\neq \pm 1$), then $M$ can be imbedded as a hypersurface.

**Proof.** In consideration of Theorem 3.1, we may assume that $M$ is imbedded as a complete intersection in $P_{n+2}(C)$, where $n=\dim M$. From Theorem 2.1, we have

$$\chi(M) = \left[\sum_{k=0}^{n} (-1)^{k} \binom{n+3}{n-k} \sigma_k \right] a_1 a_2.$$
We consider the Diophantine equation

\[(3.1) \sum_{k=0}^{n} (-1)^{k} \binom{n+3}{n-k} \sigma_k = \pm 1.\]

Multiplying both sides of (3.1) by \(a_1-a_2\), we obtain

\[\sum_{k=0}^{n} (-1)^{k} \binom{n+3}{n-k} (a_1^{k+1} - a_2^{k+1}) = \pm (a_1 - a_2),\]

which can be written as

\[\left[ \sum_{k=0}^{n+1} (-1)^{k} \binom{n+3}{n+1-k} a_1^k \right] + \left[ \sum_{k=0}^{n+1} (-1)^{k} \binom{n+3}{n+1-k} a_2^k \right] = \pm (a_1 - a_2).\]

Hence we have

\[- \frac{(1 - a_1)^{n+3} - 1 + (n + 3)a_1}{a_1^2} + \frac{(1 - a_2)^{n+3} - 1 + (n + 3)a_2}{a_2^2} = \pm (a_1 - a_2)\]
or

\[\frac{(1 - a_1)^{n+3} - 1 + (n + 3)a_1}{a_1^2} = \pm a_1\]

\[= \frac{(1 - a_2)^{n+3} - 1 + (n + 3)a_2}{a_2^2} \pm a_2.\]

Let

\[f_n(a) = \frac{(1 - a)^{n+3} - 1 + (n + 3)a}{a^2} \pm a.\]

Then it is not difficult to prove that

\[f_n'(a) > 0 \quad \text{for } a \geq 3 \quad \text{if } n \text{ is odd},\]

\[f_n'(a) < 0 \quad \text{for } a \geq 2 \quad \text{if } n \text{ is even}.\]

This implies that \(f_n(a)\) is monotonically increasing (resp. decreasing) for \(a \geq 3\) (resp. \(a \geq 2\)) if \(n\) is odd (resp. even). Therefore from (3.2) we deduce that either \(a_1 = a_2\) or \(f_n(2) = f_n(*)\), the latter case arising only when \(n\) is odd. It is easy to show that the only solution for the latter case is \(f_2(2) = f_2(3)\). But this is excluded by the assumption that \(n > 1\). Hence we have \(a_1 = a_2\).
Putting $a=a_1=a_2$, from (3.1) we have

$$
\sum_{k=0}^{n} (-1)^k (k+1) \binom{n+3}{n-k} a^k = \pm 1.
$$

Let

$$
g_n(a) = \sum_{k=0}^{n} (-1)^k (k+1) \binom{n+3}{n-k} a^k = \pm 1.
$$

Then we have $f'_n(a) = -g_n(a)$, which, together with (3.3), implies that

$$
g_n(a) < 0 \quad \text{for } a \geq 3 \quad \text{if } n \text{ is odd},
g_n(a) > 0 \quad \text{for } a \geq 2 \quad \text{if } n \text{ is even}.
$$

Therefore the only candidate for the solution for (3.4) is $(n, a) = (\text{odd}, 2)$. But, since $g_n(2) = -f'_n(2) = \mp 1$ if $n$ is odd, (3.4) has no solution of this form.

Thus we have proved that the Diophantine equation (3.1) has no solution satisfying $n>1$, $a_1 \geq 2$ and $a_2 \geq 2$. This, combined with Lemma 3.2, implies that $M$ can be imbedded as a hypersurface. Q.E.D.

We have excluded the 1-dimensional case in Theorem 3.4. The following result gives a solution for this case.

**Theorem 3.6.** Let $M$ be a complex curve which is a complete intersection manifold. If $\chi(M) = v_1 \cdots v_p$ for some prime numbers $v_1, \ldots, v_p$ ($\neq \pm 1$), then $M$ can be imbedded as a complete intersection manifold in $P_p(C)$ except when

$$
P_p(C) \ni M = M_1 \cap M_2 \quad (\deg M_1 = 2, \deg M_2 = 3),
P_p(C) \ni M = M_1 \cap M_2 \cap M_3 \quad (\deg M = 2).
$$

**Proof.** In consideration of Theorem 3.1, we may assume that $M$ is imbedded as a complete intersection manifold in $P_{1+p}(C)$. From Theorem 2.1 we have

$$
\chi(M) = [p + 2 - \sum a_a] a_1 \cdots a_p
$$

so that $\chi(M)$ is a product of $p+1$ integers. Therefore at least one of them must be equal to $\pm 1$. It is easy to prove that the equation $p+2 - \sum a_a = \pm 1$ has solutions satisfying $a_1 \geq 2, \ldots, a_p \geq 2$ only when $p=2$ or $p=3$, and the solutions are, respectively, \{a_1, a_2\} = \{2, 3\} or \{a_1, a_2, a_3\} = \{2, 2, 2\}. Q.E.D.

For complex surfaces we have the following:

**Theorem 3.7.** Let $M$ be a complex surface which is a complete intersection manifold. If $\chi(M) = v_1 \cdots v_p$ for some prime numbers $v_1, \ldots, v_p$
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Let \( M \) and \( N \) be manifolds, then \( M \) can be imbedded as a complete intersection manifold in \( P_{1+p}(C) \).

**Proof.** In consideration of Theorem 3.1, we may assume that \( M \) is imbedded as a complete intersection manifold in \( P_{2+p}(C) \). From Theorem 2.1 we have

\[
\chi(M) = \left( p + \frac{3}{2} \right) - \left( p + \frac{3}{1} \right) \sum a_x + \sum_{a_1 \leq a_2} a_{a_1} a_{a_2} \right] a_1 \cdots a_p.
\]

Let

\[
f(a_1, \cdots, a_p) = \left( p + \frac{3}{2} \right) - \left( p + \frac{3}{1} \right) \sum a_x + \sum_{a_1 \leq a_2} a_{a_1} a_{a_2}.
\]

Then it is easy to prove that \( f(a_1, \cdots, a_p) \geq (p+3)/(p+1) > 1 \). Therefore at least one of \( a_x \) must be equal to 1. Q.E.D.

**Theorem 3.8.** Let \( M \) be a 4-dimensional complete intersection manifold. If \( \chi(M) = v_1 \cdots v_p \) for some prime numbers \( v_1, \cdots, v_p \) (\( \neq 1 \)), then \( M \) can be imbedded as a complete intersection manifold in \( P_{3+p}(C) \).

**Proof.** In consideration of Theorem 3.1, we may assume that \( M \) is imbedded as a complete intersection manifold in \( P_{4+p}(C) \). From Theorem 2.1, we have

\[
\chi(M) = \left[ \sum_{k=0}^{4} (-1)^k \left( p + \frac{5}{4 - k} \right) \sigma_k \right] a_1 \cdots a_p.
\]

We consider the Diophantine equation

\[
\sum_{k=0}^{4} (-1)^k \left( p + \frac{5}{4 - k} \right) \sigma_k = 1.
\]

Let

\[
f(a_1, \cdots, a_p) = \sum_{k=0}^{4} (-1)^k \left( p + \frac{5}{4 - k} \right) \sigma_k - 1.
\]

Then we have

\[
\frac{\partial f}{\partial a_x} = -\left( p + \frac{5}{3} \right) + \left( p + \frac{5}{2} \right) (a_x + \sigma_1)
- \left( p + \frac{5}{1} \right) (a_x^2 + a_x \sigma_1 + \sigma_2) + a_x^3 + a_x^2 \sigma_1 + a_x \sigma_2 + \sigma_3.
\]

It is easy to show that \( f \) has no critical point in \( \{(a_1, \cdots, a_p) \in R^p \mid a_1 \geq 2, \cdots, a_p \geq 2\} \). We can also prove, by an induction, that \( f > 0 \) on the boundary of \( \{(a_1, \cdots, a_p) \in R^p \mid a_1 \geq 2, \cdots, a_p \geq 2\} \). These facts imply that \( f > 0 \) on \( \{(a_1, \cdots, a_p) \in R^p \mid a_1 \geq 2, \cdots, a_p \geq 2\} \). Therefore Theorem 3.8 follows from Lemma 3.2.
Bibliography


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