PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

SOME IMPLICATIONS OF THE EULER-POINCARÉ CHARACTERISTIC FOR COMPLETE INTERSECTION MANIFOLDS

BANG-YEN CHEN[1] AND KOICHI OGIUE[2]

ABSTRACT. Studies on relations between Euler-Poincaré characteristic and codimension of a complete intersection manifold in a complex projective space.

1. **Introduction.** Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space. An n-dimensional algebraic manifold M imbedded in $P_{n+p}(C)$ is called a *complete intersection manifold* if M is given as an intersection of p nonsingular hypersurfaces $M_1, \cdots, M_p$ in general position in

$$P_{n+p}(C): M = M_1 \cap \cdots \cap M_p.$$

It is known that the Chern classes of a complete intersection manifold M are completely determined by the degrees of $M_1, \cdots, M_p$. In particular, the Euler-Poincaré characteristic of a complete intersection manifold is completely determined by the degrees of $M_1, \cdots, M_p$.

In §2 we prove a formula for the Euler-Poincaré characteristic of a complete intersection manifold in terms of the degrees.

It is sometimes very important to know the smallest codimension for a complete intersection manifold: the smallest p for which M can be imbedded as a complete intersection manifold in $P_{n+p}(C)$. In §3 we prove several results in this direction.

2. **The Euler-Poincaré characteristic of complete intersection manifolds.** Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space and let M be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C): M=M_1\cap\cdots\cap M_p$, where the M_α's are nonsingular hypersurfaces in $P_{n+p}(C)$. The following theorem gives a concrete formula for the Euler-Poincaré characteristic of a complete intersection manifold.

Received by the editors April 17, 1973 and, in revised form, August 31, 1973.
AMS (MOS) subject classifications (1970). Primary 14M10, 57D20.
Key words and phrases. Euler-Poincaré characteristic, complete intersection manifold, complex projective space, codimension.
[1] Work done under partial support by NSF Grant 36684.
[2] Work done under partial support by the Matsunaga Science Foundation.

THEOREM 2.1.[3] *Let $M=M_1\cap\cdots\cap M_p$ be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. If* $\deg M_\alpha=a_\alpha$, *then the Euler-Poincaré characteristic $\chi(M)$ of M is given by*

$$\chi(M)=\left[\sum_{k=0}^{n}(-1)^k\binom{n+p+1}{n-k}\sigma_k\right]\left(\prod_{\alpha=1}^{p}a_\alpha\right),$$

where $\sigma_k=\sum_{\alpha_1\leqq\cdots\leqq\alpha_k}a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_k}$ (the sum of all homogeneous monomials of degree k in $a_1, a_2, \cdots, a_p$) and $\binom{n+p+1}{n-k}$ is the binomial coefficient.

PROOF. Let $\tilde{h}$ be the generator of $H^2(P_{n+p}(C), Z)$ corresponding to the divisor class of a hyperplane $P_{n+p-1}(C)$. Then the total Chern class $c(P_{n+p}(C))$ of $P_{n+p}(C)$ is given by

$$c(P_{n+p}(C))=(1+\tilde{h})^{n+p+1}$$

Let $j:M\to P_{n+p}(C)$ be the imbedding and ν be the normal bundle of $j(M)$ in $P_{n+p}(C)$. Then the total Chern class $c(\nu)$ of ν is given by

$$c(\nu)=(1+a_1h)\cdots(1+a_ph),$$

where h is the image of $\tilde{h}$ under the homomorphism $j^*:H^2(P_{n+p}(C), Z)\to H^2(M, Z)$. Since $j^*T(P_{n+p}(C))=T(M)\oplus\nu$ (Whitney sum), we have

$$j^*c(P_{n+p}(C))=c(M)\cdot c(\nu),$$

where $c(M)$ is the total Chern class of M. Let $c_i(M)$ be the ith Chern class of M. Then we have

$$(1+h)^{n+p+1}=[1+c_1(M)+\cdots+c_n(M)]\cdot(1+a_1h)\cdots(1+a_ph)$$

which implies that

$$c_n(M)=\left[\sum_{k=0}^{n}(-1)^k\binom{n+p+1}{n-k}\sigma_k\right]h^n.$$

Taking the values of both sides on the fundamental cycle of M, we have

$$\chi(M)=\left[\sum_{k=0}^{n}(-1)^k\binom{n+p+1}{n-k}\sigma_k\right]\left(\prod_{\alpha=1}^{p}a_\alpha\right).$$

REMARK. Let $b_i(M)$ be the ith Betti number of M. It is known that if M is an n-dimensional complete intersection manifold, then

$$b_{2k}(M)=1 \qquad (2k\neq n),$$
$$b_{2k+1}(M)=0 \qquad (2k+1\neq n).$$

[3] Although Theorem 2.1 can be obtained from Theorem 22.1.1 of [1], we give here a direct proof for the sake of completeness.

Therefore $\chi(M)=\sum_{i=0}^{2n}(-1)^i b_i(M) \geqq n+1$ (resp. $\leqq n+1$) provided that n is even (resp. odd).

3. **Some implications of the Euler-Poincaré characteristic.** First we prove the following:

THEOREM 3.1. *Let M be an n-dimensional complete intersection manifold. If $\chi(M)=\nu_1\cdots\nu_p$ for some prime numbers $\nu_1,\cdots,\nu_p$ $(\neq\pm 1)$, then M can be imbedded as a complete intersection manifold in $P_{n+p}(C)$.*

PROOF. We can assume without loss of generality that M is a complete intersection manifold imbedded in $P_{n+q}(C)$ for some $q\geqq p$. In fact, if M is a complete intersection manifold imbedded in $P_{n+r}(C)$ for some $r<p$, then, by imbedding $P_{n+r}(C)$ into $P_{n+q}(C)$ as a linear subspace for some $q\geqq p$, M can be considered as a complete intersection manifold imbedded in $P_{n+q}(C)$.

Theorem 2.1 implies that $\chi(M)$ is given as a product of $q+1$ integers: $\chi(M)=[\cdots]a_1\cdots a_q$. On the other hand, since $\chi(M)=\nu_1\cdots\nu_p$, at least $q-p$ a_α's must be equal to 1. This implies that M can be imbedded as a complete intersection manifold in some $(n+p)$-dimensional linear subspace $P_{n+p}(C)$ in $P_{n+q}(C)$. Q.E.D.

It is sometimes very important to know the smallest codimension for which M can be imbedded as a complete intersection manifold. From this point of view, Theorem 3.1 may not be the best possible in general. We shall prove several partial improvements of Theorem 3.1 in the following: First we prove the following:

LEMMA 3.2. *Let M be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. Assume that $\chi(M)=\nu_1\cdots\nu_p$ for some prime numbers $\nu_1,\cdots,\nu_p$ $(\neq\pm 1)$. If the Diophantine equation $\sum_{k=0}^{n}(-1)^k\binom{n+p+1}{n-k}\sigma_k=\pm 1$ has no solution satisfying $a_1\geqq 2,\cdots,a_p\geqq 2$, then M can be imbedded as a complete intersection manifold in $P_{n+p-1}(C)$.*

PROOF. Theorem 2.1 implies that $\chi(M)$ is given as a product of $p+1$ integers: $\chi(M)=[\cdots]a_1\cdots a_p$. On the other hand, since $\chi(M)=\nu_1\cdots\nu_p$, at least one among $[\cdots]$, $a_1,\cdots,a_p$ must be equal to ± 1. If the Diophantine equation $[\cdots]=\pm 1$ has no solution satisfying $a_1>1,\cdots,a_p>1$, then at least one a_α must be equal to 1. Q.E.D.

THEOREM 3.3. *Let M be a complete intersection manifold. If $\chi(M)$ is a prime number, then M is one of the following:*

(1) *a linear subspace*

(2) *a quadric in $P_2(C)$,*

(3) *a curve of degree 4 in $P_2(C)$.*

PROOF. In consideration of Theorem 3.1, we may assume that M is a nonsingular hypersurface of $P_{n+1}(C)$. Putting $a=a_1$, from Theorem 2.1 we have

$$\chi(M) = \left[\sum_{k=0}^{n}(-1)^k\binom{n+2}{n-k}a^k\right]a = \frac{(1-a)^{n+2}-1+(n+2)a}{a^2}\cdot a.$$

Since $\chi(M)$ is a prime number, either $a=1$ or

$$\frac{(1-a)^{n+2}-1+(n+2)a}{a^2} = \pm 1.$$

If $a=1$, then M is a linear subspace. Therefore we consider the latter case. It suffices to prove the following:

LEMMA. *The only positive integral solutions for the Diophantine equation* $(1-a)^{n+2}-1+(n+2)a=\pm a^2$ *are* $(n, a)=(1, 2)$ *and* $(n, a)=(1, 4)$.

PROOF OF LEMMA. Since $a=1$ cannot be a solution for the equation $(1-a)^{n+2}-1+(n+2)a=\pm a^2$, we consider this equation for $a\geqq 2$. Let

$$f_n(a) = \frac{(1-a)^{n+2}-1+(n+2)a}{a^2} \quad \text{for } a \geqq 2.$$

(i) If n is even: It is easy to show that $f_n(a)$ is monotonically increasing and $f_n(2)=(n+2)/2>1$. Therefore the equation $(1-a)^{n+2}-1+(n+2)a=\pm a^2$ has no solution of the form $(n, a)=(\text{even}, *)$.

(ii) If n is odd: $f_n(a)$ is monotonically decreasing and $f_n(2)=(n+1)/2$. It is also easy to show that $f_n(5)\leqq -2$. Therefore the only candidates for the solutions for the equation $(1-a)^{n+2}-1+(n+2)a=\pm a^2$ are of the form $(n, a)=(*, 2)$, $(n, a)=(*, 3)$ or $(n, a)=(*, 4)$. We can easily prove that the only solutions are $(n, a)=(1, 2)$ and $(n, a)=(1, 4)$. Q.E.D.

COROLLARY 3.4. *Let M be a complete intersection manifold. If* $\dim M>1$ *and if $\chi(M)$ is a prime number, then M is a linear subspace.*

THEOREM 3.5. *Let M be a complete intersection manifold. If* $\dim M>1$ *and if* $\chi(M)=\nu_1\nu_2$ *for some prime numbers* ν_1, ν_2 $(\neq\pm 1)$, *then M can be imbedded as a hypersurface.*

PROOF. In consideration of Theorem 3.1, we may assume that M is imbedded as a complete intersection in $P_{n+2}(C)$, where $n=\dim M$. From Theorem 2.1, we have

$$\chi(M) = \left[\sum_{k=0}^{n}(-1)^k\binom{n+3}{n-k}\sigma_k\right]a_1a_2.$$

We consider the Diophantine equation

$$\sum_{k=0}^{n}(-1)^k\binom{n+3}{n-k}\sigma_k = \pm 1. \tag{3.1}$$

Multiplying both sides of (3.1) by a_1-a_2, we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n+3}{n-k}(a_1^{k+1}-a_2^{k+1}) = \pm(a_1-a_2),$$

which can be written as

$$-\left[\sum_{k=0}^{n+1}(-1)^k\binom{n+3}{n+1-k}a_1^k\right] + \left[\sum_{k=0}^{n+1}(-1)^k\binom{n+3}{n+1-k}a_2^k\right] = \pm(a_1-a_2).$$

Hence we have

$$-\frac{(1-a_1)^{n+3}-1+(n+3)a_1}{a_1^2} + \frac{(1-a_2)^{n+3}-1+(n+3)a_2}{a_2^2} = \pm(a_1-a_2)$$

or

$$\frac{(1-a_1)^{n+3}-1+(n+3)a_1}{a_1^2} \pm a_1 = \frac{(1-a_2)^{n+3}-1+(n+3)a_2}{a_2^2} \pm a_2. \tag{3.2}$$

Let

$$f_n(a) = \frac{(1-a)^{n+3}-1+(n+3)a}{a^2} \pm a.$$

Then it is not difficult to prove that

$$\begin{aligned} &f'_n(a) > 0 \quad \text{for } a \geqq 3 \quad \text{if } n \text{ is odd,} \\ &f'_n(a) < 0 \quad \text{for } a \geqq 2 \quad \text{if } n \text{ is even.} \end{aligned} \tag{3.3}$$

This implies that $f_n(a)$ is monotonically increasing (resp. decreasing) for $a\geqq 3$ (resp. $a\geqq 2$) if n is odd (resp. even). Therefore from (3.2) we deduce that either $a_1=a_2$ or $f_n(2)=f_n(*)$, the latter case arising only when n is odd. It is easy to show that the only solution for the latter case is $f_1(2)=f_1(3)$. But this is excluded by the assumption that $n>1$. Hence we have $a_1=a_2$.

Putting $a=a_1=a_2$, from (3.1) we have

$$\sum_{k=0}^{n}(-1)^k(k+1)\binom{n+3}{n-k}a^k=\pm 1. \tag{3.4}$$

Let

$$g_n(a)=\sum_{k=0}^{n}(-1)^k(k+1)\binom{n+3}{n-k}a^k\mp 1.$$

Then we have $f'_n(a)=-g_n(a)$, which, together with (3.3), implies that

$$g_n(a)<0 \quad \text{for } a\geqq 3 \quad \text{if } n \text{ is odd,}$$
$$g_n(a)>0 \quad \text{for } a\geqq 2 \quad \text{if } n \text{ is even.}$$

Therefore the only candidate for the solution for (3.4) is $(n, a)=(\text{odd}, 2)$. But, since $g_n(2)=-f'_n(2)=\mp 1$ if n is odd, (3.4) has no solution of this form.

Thus we have proved that the Diophantine equation (3.1) has no solution satisfying $n>1$, $a_1\geqq 2$ and $a_2\geqq 2$. This, combined with Lemma 3.2, implies that M can be imbedded as a hypersurface. Q.E.D.

We have excluded the 1-dimensional case in Theorem 3.4. The following result gives a solution for this case.

THEOREM 3.6. *Let M be a complex curve which is a complete intersection manifold. If $\chi(M)=\nu_1\cdots\nu_p$ for some prime numbers $\nu_1,\cdots,\nu_p$ $(\neq\pm 1)$, then M can be imbedded as a complete intersection manifold in $P_p(C)$ except when*

$$P_3(C)\supset M=M_1\cap M_2 \quad (\deg M_1=2, \deg M_2=3),$$
$$P_4(C)\supset M=M_1\cap M_2\cap M_3 \quad (\deg M_\alpha=2).$$

PROOF. In consideration of Theorem 3.1, we may assume that M is imbedded as a complete intersection manifold in $P_{1+p}(C)$. From Theorem 2.1 we have

$$\chi(M)=\Big[p+2-\sum a_\alpha\Big]a_1\cdots a_p$$

so that $\chi(M)$ is a product of $p+1$ integers. Therefore at least one of them must be equal to ± 1. It is easy to prove that the equation $p+2-\sum a_\alpha=\pm 1$ has solutions satisfying $a_1\geqq 2,\cdots,a_p\geqq 2$ only when $p=2$ or $p=3$, and the solutions are, respectively, $\{a_1, a_2\}=\{2, 3\}$ or $\{a_1, a_2, a_3\}=\{2, 2, 2\}$. Q.E.D.

For complex surfaces we have the following:

THEOREM 3.7. *Let M be a complex surface which is a complete intersection manifold. If $\chi(M)=\nu_1\cdots\nu_p$ for some prime numbers $\nu_1,\cdots,\nu_p$*

($\neq 1$), *then M can be imbedded as a complete intersection manifold in* $P_{1+p}(C)$.

PROOF. In consideration of Theorem 3.1, we may assume that M is imbedded as a complete intersection manifold in $P_{2+p}(C)$. From Theorem 2.1 we have

$$\chi(M) = \left[\binom{p+3}{2} - \binom{p+3}{1}\sum a_\alpha + \sum_{\alpha_1 \leqq \alpha_2} a_{\alpha_1} a_{\alpha_2}\right] a_1 \cdots a_p.$$

Let

$$f(a_1, \cdots, a_p) = \binom{p+3}{2} - \binom{p+3}{1}\sum a_\alpha + \sum_{\alpha_1 \leqq \alpha_2} a_{\alpha_1} a_{\alpha_2}.$$

Then it is easy to prove that $f(a_1, \cdots, a_p) \geqq (p+3)/(p+1) > 1$. Therefore at least one of a_α must be equal to 1. Q.E.D.

THEOREM 3.8. *Let M be a 4-dimensional complete intersection manifold. If $\chi(M) = \nu_1 \cdots \nu_p$ for some prime numbers $\nu_1, \cdots, \nu_p$ ($\neq 1$), then M can be imbedded as a complete intersection manifold in $P_{3+p}(C)$.*

PROOF. In consideration of Theorem 3.1, we may assume that M is imbedded as a complete intersection manifold in $P_{4+p}(C)$. From Theorem 2.1, we have

$$\chi(M) = \left[\sum_{k=0}^{4} (-1)^k \binom{p+5}{4-k} \sigma_k\right] a_1 \cdots a_p.$$

We consider the Diophantine equation

$$\sum_{k=0}^{4} (-1)^k \binom{p+5}{4-k} \sigma_k = 1. \tag{3.5}$$

Let

$$f(a_1, \cdots, a_p) = \sum_{k=0}^{4} (-1)^k \binom{p+5}{4-k} \sigma_k - 1.$$

Then we have

$$\frac{\partial f}{\partial a_\alpha} = -\binom{p+5}{3} + \binom{p+5}{2}(a_\alpha + \sigma_1)$$
$$- \binom{p+5}{1}(a_\alpha^2 + a_\alpha\sigma_1 + \sigma_2) + a_\alpha^3 + a_\alpha^2\sigma_1 + a_\alpha\sigma_2 + \sigma_3.$$

It is easy to show that f has no critical point in $\{(a_1, \cdots, a_p) \in R^p | a_1 \geqq 2, \cdots, a_p \geqq 2\}$. We can also prove, by an induction, that $f > 0$ on the boundary of $\{(a_1, \cdots, a_p) \in R^p | a_1 \geqq 2, \cdots, a_p \geqq 2\}$. These facts imply that $f > 0$ on $\{(a_1, \cdots, a_p) \in R^p | a_1 \geqq 2, \cdots, a_p \geqq 2\}$. Therefore Theorem 3.8 follows from Lemma 3.2.

BIBLIOGRAPHY

1. F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Ergebnisse der Math. und ihrer Grenzgebiete, Heft 9, Springer-Verlag, Berlin, 1956; English transl., Die Grundlehren der math. Wissenschaften, Band 131, Springer-Verlag, New York, 1966. MR **18**, 509; **34** #2573.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823