

THE RING OF HOLOMORPHIC FUNCTIONS ON A STEIN COMPACT SET AS A UNIQUE FACTORIZATION DOMAIN

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ABSTRACT. Let Γ be the ring of germs of analytic functions on a Stein compact subset K of a complex-analytic space. Necessary and sufficient conditions on K for Γ to be a unique factorization domain are given.

Let X be a complex-analytic space with structure sheaf \mathcal{O} , and let K be a compact Stein subset of X , so that K has a neighbourhood base in X consisting of Stein open subsets of X . Let $\Gamma = \Gamma(K, \mathcal{O})$, the ring of germs of analytic functions on K . In this note, we determine necessary and sufficient conditions for Γ to be a unique factorization domain.

Conditions for Γ to be a Noetherian ring have been given by Siu [5], following an earlier result of Frisch [2]:

THEOREM (SIU). *Let K be a compact Stein subset of an analytic space (X, \mathcal{O}) . Then $\Gamma(K, \mathcal{O})$ is Noetherian if and only if $V \cap K$ has only finitely many topological components for each complex-analytic subvariety V defined in an open neighbourhood of K .*

If z is a point of an arbitrary complex-analytic space, then \mathcal{O}_z , the ring of analytic function germs at z , is not necessarily a unique factorization domain, for, if it is, then z is a normal point of X . On the other hand, if z is a simple point of X , which means that \mathcal{O}_z is a regular local ring, or, equivalently, that a neighbourhood of z in X can be mapped bianalytically onto an open set in \mathbb{C}^n , then \mathcal{O}_z is a unique factorization domain by [4, §4.2]. We shall suppose explicitly that \mathcal{O}_z is a unique factorization domain for each z in K . Thus, the hypotheses on K are satisfied if K is a compact subset of a complex manifold X and K is the intersection of a sequence of open Stein manifolds (i.e., K is a holomorphic set).

We shall require Cartan's theorems A and B for coherent analytic sheaves over the compact set K . These theorems are stated for the context in which we require them in [1, Théorème 1].

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Let \mathcal{M} denote the sheaf of meromorphic functions on K , and let \mathcal{O}^* denote the sheaf of invertible elements in \mathcal{O} (with multiplication as group operation). We say that K is a *Cousin II set* if the second Cousin problem can be solved on K : if $\{U_\alpha\}$ is a cover of K consisting of (relatively) open subsets, and if $f_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ such that $f_\alpha f_\beta^{-1} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*)$ for each α, β , then there exists $f \in \Gamma(K, \mathcal{O})$ such that $ff_\alpha^{-1} \in \Gamma(U_\alpha, \mathcal{O}^*)$ for each α . The set K is a *strong Poincaré set* if, given $m \in \Gamma(K, \mathcal{M})$, there exist $f, g \in \Gamma(K, \mathcal{O})$ such that $m = fg^{-1}$ and $(f_z, g_z) = 1_z$ in \mathcal{O}_z ($z \in K$) (i.e., the germs f_z and g_z are coprime for each z). Let I be an ideal in $\Gamma(K, \mathcal{O})$, and let \mathcal{I} be the sheaf of ideals of $\mathcal{O}|K$ generated over K by I . Then we say that I is a *locally principal ideal* in Γ if \mathcal{I}_z is a principal ideal in \mathcal{O}_z for each $z \in K$.

THEOREM 1. *Let K be a compact Stein subset of a complex-analytic space X with structure sheaf \mathcal{O} , and suppose \mathcal{O}_z is a unique factorization domain for each $z \in K$. Suppose that $\Gamma = \Gamma(K, \mathcal{O})$ is a Noetherian domain. Then the following conditions are equivalent:*

- (1) Γ is a unique factorization domain;
- (2) $H^2(K; \mathbb{Z}) = \{0\}$;
- (3) K is a Cousin II set;
- (4) K is a strong Poincaré set;
- (5) every locally principal ideal in Γ is a principal ideal.

PROOF. We first prove the implications which do not require the supposition that Γ be a Noetherian ring.

(2) \Leftrightarrow (3) \Rightarrow (4). Using Cartan's Theorems A and B, these are standard results (cf. [3, VIII, B]).

(5) \Rightarrow (3). Let $\{f_\alpha; U_\alpha\}$ be data for the second Cousin problem on K . Define the sheaf \mathcal{F} over K by the conditions $\mathcal{F}_z = \langle f_{\alpha,z} \rangle$ ($z \in U_\alpha$). Then \mathcal{F}_z is well defined, for $\langle f_{\alpha,z} \rangle = \langle f_{\beta,z} \rangle$ whenever $z \in U_\alpha \cap U_\beta$, and \mathcal{F} is a coherent analytic sheaf over K . Let $I = \Gamma(K, \mathcal{F})$ be the algebra of sections of \mathcal{F} over K , and let I generate \mathcal{I} in $\mathcal{O}|K$. By Theorem A, $\mathcal{I}_z = \mathcal{F}_z$ for $z \in K$, so that I is a locally principal ideal in Γ . By hypothesis, I is a principal ideal in Γ , say $I = \langle f \rangle$. Clearly, f is a solution of the problem with data $\{f_\alpha; U_\alpha\}$.

(1) \Rightarrow (4). Let $m \in \Gamma(K, \mathcal{M})$. Since K is a Stein set, we can write $m = f_1 g_1^{-1}$ with $f_1, g_1 \in \Gamma$ (cf. [3, VIII, B, 10]), and, by Theorem A again, we can suppose that $(f_{1,z}, g_{1,z}) = 1_z$ for some fixed $z \in K$. Clearly, using the hypothesis, we can also suppose that $(f_1, g_1) = 1$ in Γ . Let w be any point of K , and write $m = f_2 g_2^{-1}$ with $f_2, g_2 \in \Gamma$, $(f_2, g_2) = 1$ in Γ , and $(f_{2,w}, g_{2,w}) = 1_w$. Then $f_1 g_2 = f_2 g_1$ in Γ , and, since Γ is a unique factorization domain, $f_1 \sim f_2$ and $g_1 \sim g_2$ in Γ . Thus, $(f_{1,w}, g_{1,w}) = (f_{2,w}, g_{2,w}) = 1_w$ in \mathcal{O}_w for each

$w \in K$. It follows that f_1 and g_1 are coprime at each point of K , and we have written $m=f_1g_1^{-1}$ in the required form.

In the proof of the remaining implications, we use the fact that Γ is a Noetherian domain.

(4) \Rightarrow (1). Since Γ is Noetherian, it is sufficient to prove that every irreducible element in Γ is prime. Let f be an irreducible in Γ , and suppose that $gh \in \langle f \rangle$, say $gh=ff_1$ in Γ . Since K is a strong Poincaré set and Γ is an integral domain, we may suppose that $(g_z, f_{1,z})=1_z$ in \mathcal{O}_z ($z \in K$). Since $g_z h_z=f_z f_{1,z}$, $h_z \in \langle f_{1,z} \rangle$ ($z \in K$), and so, from Theorem B, $h \in \langle f_1 \rangle$ in Γ , say $h=f_1 h_1$. Thus, $gh_1=f$, and since f is irreducible, either g or h_1 is a unit in Γ . In the former case, $h \in \langle f \rangle$, and in the latter, $g \in \langle f \rangle$, so that f is a prime, as required.

(1) \Rightarrow (5). Let I be a locally principal ideal in Γ , and let \mathcal{S} be as above. Since I is finitely generated, \mathcal{S} is a coherent analytic sheaf over K .

Let $z \in K$. By hypothesis, \mathcal{S}_z is a principal ideal in \mathcal{O}_z , and so, by Theorem A, there exists $f \in I$ such that $\mathcal{S}_z=\langle f_z \rangle$ in \mathcal{O}_z . Since K is compact, there exist $f_1, \dots, f_k \in I$ such that, for each $z \in K$, $\mathcal{S}_z=\langle f_{i,z} \rangle$ for some $i=i(z) \in \{1, \dots, k\}$. By Theorem B, $I=\langle f_1, \dots, f_k \rangle$. Since Γ is a unique factorization domain, f_1, \dots, f_k have a highest common factor in Γ , say $g=(f_1, \dots, f_k)$. Let $h_i=f_i g^{-1}$ ($i=1, \dots, k$), so that $(h_1, \dots, h_k)=1$.

We now use hypothesis (1) to prove that, if $(p_1, \dots, p_n)=1$ in Γ , then $(p_{1,z}, \dots, p_{n,z})=1_z$ in \mathcal{O}_z ($z \in K$). The result holds for the case $n=2$ by the result '(1) \Rightarrow (4)', above, and the general result follows by an immediate inductive argument. Thus, we see that we have $h_1, \dots, h_k \in \Gamma$ with

$$(*) \quad (h_{1,z}, \dots, h_{k,z}) = 1_z \quad \text{in } \mathcal{O}_z \quad (z \in K).$$

Take $z \in K$, and suppose that $\mathcal{S}_z=\langle f_{j,z} \rangle$. If $i \in \{1, \dots, k\}$, there exists $p_{i,z} \in \mathcal{O}_z$ such that $f_{i,z}=p_{i,z} f_{j,z}$. Thus, $h_{i,z}=p_{i,z} h_{j,z}$ and so $h_{j,z} | h_{i,z}$ in \mathcal{O}_z ($i \in \{1, \dots, k\}$). This shows that $h_{j,z} | (h_{1,z}, \dots, h_{k,z})$ in \mathcal{O}_z . From (*), $h_{j,z}$ is a unit in \mathcal{O}_z , and we have proved that the functions h_1, \dots, h_k have no common zero on K . It is a consequence of Cartan's Theorem B [3, VIII, A, 16] that there exist $p_1, \dots, p_k \in \Gamma$ such that $\sum p_i h_i=1$. Hence, $g=\sum p_i g f_i \in I$, $I=\langle g \rangle$, and I is a principal ideal in Γ , as required.

This concludes the proof of the theorem. \square

Suppose that Γ is a regular Noetherian domain in the sense of Kaplansky [4]. Then the following proof that '(5) \Rightarrow (1)' holds.

Let I be an ideal in Γ generating the coherent sheaf \mathcal{S} in $\mathcal{O}|K$. Suppose that I is invertible [4, p. 37] in Γ . We first note that \mathcal{S}_z is invertible in \mathcal{O}_z ($z \in K$). We must prove that $\mathcal{O}_z \subset \mathcal{S}_z \mathcal{S}_z^{-1}$, so take $f_z \in \mathcal{O}_z$. Then $f_z \in \sum f_{i,z} \mathcal{O}_z$ with $f_i \in \Gamma$. Since I is invertible, there exist $g_{ij} \in I$ and $h_{ij} \in I^{-1}$ such that $f_i=\sum_j g_{ij} h_{ij}$ for each i . Thus, $f_z \in \sum_{i,j} g_{ij,z} h_{ij,z} \mathcal{O}_z$. Clearly, $g_{ij,z} \in \mathcal{S}_z$. Also, $h_{ij,z} \in \mathcal{S}_z^{-1}$, for, if $p_z \in \mathcal{S}_z$, then, by Theorem A,

$p_z \in \sum q_{k,z} \mathcal{O}_z$ with $q_k \in I$, and, since I is invertible, $h_{ij}(\sum q_k \Gamma) \subset \Gamma$, so that $h_{ij,z} p_z \in \mathcal{O}_z$ and $h_{ij,z} \in \mathcal{I}_z^{-1}$, as required. Thus, $\mathcal{O}_z \subset \mathcal{I}_z \mathcal{I}_z^{-1}$.

Now, by [4, Theorem 60], \mathcal{I}_z is principal in \mathcal{O}_z ($z \in K$), so, by hypothesis, I is principal in Γ , and the result follows by [4, Theorem 185]. \square

If K is a compact, holomorphic set in C^n for which Γ is a Noetherian domain, then Γ is a regular Noetherian domain. To show this, it suffices to show that every maximal ideal of Γ can be generated by an R -sequence [4, §3.1]. But if M is a maximal ideal of Γ , then $M = \{f \in \Gamma : f(z^0) = 0\}$ for some $z^0 \in K$, and it is clear that the elements $z_1 - z_1^0, \dots, z_n - z_n^0$ form the required R -sequence.

COROLLARY. *Let Δ be a compact polydisc in C^n . Then $\Gamma(\Delta, \mathcal{O})$ is a unique factorization domain.*

PROOF. That $\Gamma(\Delta, \mathcal{O})$ is a Noetherian domain is noted in [2]. Certainly, $H^2(\Delta; \mathbf{Z}) = \{0\}$.

When $\Gamma = \Gamma(K, \mathcal{O})$ is Noetherian, we have the following Nullstellensatz for Γ . We use the notation of [3, II, E].

PROPOSITION 2. *Let K be a compact Stein subset of a complex-analytic space X , and suppose that Γ is a Noetherian domain. Let I be an ideal in Γ . Then $\text{id loc } I = \text{rad } I$.*

PROOF. Certainly, $\text{rad } I \subset \text{id loc } I$. Take $f \in \text{id loc } I$. If $z \in K$, then there exists $k(z) \in N$ such that $f_z^{k(z)} \in \mathcal{I}_z$. This is the standard Nullstellensatz, proved in the local complex-analytic case in [3, III. A. 7]; the general case follows by writing \mathcal{I} locally as a quotient of an ideal sheaf in C^n [6, Lemma]. For w in a neighbourhood of z , $f_w^{k(z)} \in \mathcal{I}_w$. Take a finite refinement of the neighbourhoods covering K , corresponding to z_1, \dots, z_m , and let $k = \max\{k(z_1), \dots, k(z_m)\}$. Then $f_z^k \in \mathcal{I}_z$ ($z \in K$), and, since \mathcal{I} is coherent, $f^k \in I$. Thus, $\text{id loc } I \subset \text{rad } I$, as required. \square

Let $f \in \Gamma$, and write $V(f) = \{z \in K : f(z) = 0\}$. A variety V is irreducible if $V = V_1 \cup V_2$, where V_1 and V_2 are varieties, implies that either $V = V_1$ or $V = V_2$.

THEOREM 3. *Let K and X be as in Theorem 1. Suppose that Γ is a Noetherian domain which is a unique factorization domain, and let $f \in \Gamma$.*

(i) *$V(f)$ is an irreducible variety if and only if $f = g^n$, where g is irreducible in Γ .*

(ii) *If $f = \prod_{i=1}^n f_i^{k_i}$ is the factorization of f into irreducible factors in Γ , then $V(f) = \bigcup_{i=1}^n V(f_i)$ is the decomposition of V into its irreducible branches.*

PROOF. If $f \in \Gamma$ is irreducible, then $\langle f \rangle$ is prime, and so, by Proposition 2, $\text{id } V(f)$ is prime. The results now follow by the same arguments as those of [3, II, E].

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