ON A MULTIPLICATION DECOMPOSITION THEOREM
IN A DEDEKIND $\sigma$-COMPLETE PARTIALLY
ORDERED LINEAR ALGEBRA

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Abstract. Suppose a Dedekind $\sigma$-complete partially ordered linear algebra (dsc-pola) satisfies a certain multiplication decomposition property (see definition below), then we show that this partially ordered linear algebra actually has the same structure of a special class of real matrix algebras, consisting of elements that can be decomposed as diagonal part plus nilpotent part $w$, such that $w^2 = 0$.

A dsc-pola, denoted by $A$ (or $B$) is a real linear associative algebra which satisfies the following two conditions: (1) It is partially ordered so that it is a directed partially ordered linear space and $0 \leq xy$ whenever $x, y \in A$, $0 \leq x, 0 \leq y$. (2) It is Dedekind $\sigma$-complete, i.e., if $x_n \in A$, $0 \leq \cdots \leq x_0 \leq x_1$, then $\inf\{x_n\}$ exists. A dsc-pola $A$ has the Archimedean property: If $x, y \in A$ and $nx \leq y$ for every positive integer $n$, then $x \leq 0$. In this paper we will assume $A$ has a multiplicative identity $1^A$. Let $I = \{y : y^2 = 1, \text{ and } y^1 = 0\} \subset A$. Define $A_1 = \bigcup_{y \in I} \{x : -y \leq x \leq y\}$. Then it was shown by R. DeMarr that the multiplication of the elements in $A_1$ is commutative, and $A_1$ behaves much like an algebra of real-valued functions; moreover, $A_1$ is a lattice and has no nonzero nilpotent. For the details of the proofs and examples of $A_1$ we refer to [2]. (Note in [2], instead of the term dsc-pola, we use polac; actually they have the same meaning.) We will call $A_1$ the functional or diagonal part of $A$. Let $A$ be a dsc-pola which has the following multiplication decomposition property (abbreviated as MD):

MD property: If $y_1, y_2 \in A$, $0 \leq y_1, 0 \leq y_2$, $0 \leq u \leq y_1 y_2$, then there exists $u_i \in A$, $0 \leq u_i \leq y_i (i = 1, 2)$ such that $u = u_1 u_2$.

It was shown as Theorem 4 in [4] that if $A$ is commutative and has the MD property, then $A = A_1$. In this paper we will drop the commutativity assumption and show the following theorem:

Main Theorem. If a dsc-pola $A$ has the MD property, then for each $x \in A$, $x = d + v$, where $d \in A_1$, $v^2 = 0$, and this expression is unique (in the sense that if $x = d + v = e + u$, $e \in A_1$, $u^2 = 0$, then $d = e, v = u$).
Lemma 1. For any dsc-pola $B$ if $w \in B$, $w^2 = 0$ and $w \geq -1$, then $w \geq 0$.

Proof. Since $1 + w \geq 0$, we have $(1 + w)^n \geq 0$ or $1 + nw \geq 0$, for all $n > 0$. This means $w \geq -(1/n)1$ for all $n$. By the Archimedean property we have $w \geq 0$. □

Lemma 2. For any dsc-pola $B$ if $w \geq 0$, $w^2 = 0$, then for any $0 \leq a \in B_1$ ($B_1$ is the diagonal part of $B$), $(aw)^2 = (wa)^2 = 0$.

Proof. See the remark of Theorem II. 3.6 of [2]. □

Lemma 3. If a dsc-pola $B$ has the following property: given any $1 \leq x \in B$, $x^{-1}$ exists and $x^{-1} \leq 1$, then for any $0 \leq w \in B$, $w^n = 0$, $n \geq 2$, we have $w^a = 0$; moreover, the sum (product) of positive nilpotents is a nilpotent (zero).

Proof. See Theorems II. 3.1, II. 3.2 and its corollary in [2]. □

Lemma 4. For any dsc-pola $B$, let $x \in B$, $0 \leq x \leq 1$, if there exists $0 \leq y$ such that $1 \leq xy + yx$, then $x^{-1}$ exists and $x^{-1} \geq 1$.

Proof. Put $z = 1 - x \geq 0$. By assumption we have $1 \leq xy + yx = (1 - z)y + y(1 - z)$ or $1 \leq 1 + zy + yz \leq 2y$. Hence, $2y \geq 1 + z(1) + (1)z = 1 + z$. By induction we will show $2y \geq 1 + \sum_{k=1}^{n} z^k = h_n$ for all $n$. The assertion is clearly true for $n = 1$. If the assertion is true for $n = m$, i.e., $2y \geq h_m$, then for $n = m + 1$, we first observe that $2yz \geq h_m z$, $2zy \geq zh_m$ and $h_m z = zh_m$; hence,

$$2y \geq 1 + yz + zy \geq 1 + \frac{1}{2}(h_m z + zh_m) = 1 + zh_m = 1 + \sum_{k=1}^{m+1} z^k = h_{m+1}.$$

Therefore, $h_n$ is bounded above by $2y$, by Proposition 2 in [3] we see

$$1 \leq h = \sup \{h_n\} = \sum_{k=0}^{\infty} z^k = (1 - z)^{-1} \leq 2y.$$

Theorem 5. Let the dsc-pola $A$ have the MD property. If $0 \leq x \in A$, then $x = c + w$, where $0 \leq c \in A_1$, $0 \leq w$ and $w^a = 0$.

Proof. Put $y = x + 2 \geq 2$. Clearly $1 \leq y^2 - 1 \leq y^2$. By the MD property there exists $z_1, z_2 \in A$ such that $0 \leq z_1 \leq y$, $0 \leq z_2 \leq y$ and $y^2 - 1 = z_1 z_2$. Thus

$$1 = y(y - z_2) + (y - z_1)z_2 = (y - z_1)y + z_1(y - z_2).$$

From this we see easily that

$$1 \geq y(y - z_2) \geq y - z_2 \geq 0, \quad 1 \geq (y - z_1)y \geq y - z_1 \geq 0.$$

Hence, $z_1 \geq y - 1 \geq 1$, $z_2 \geq y - 1 \geq 1$. Put $a = y - z_1$, $b = y - z_2$. Then $1 \geq ay \geq a \geq 0, 1 \geq yb \geq b \geq 0$; this means $a, b, ay, yb$ all belong to $A_1$; therefore, they
commute with each other. Now $0 \leq a + b \leq 1 = az_2 + yb \leq ay + yb \leq 2$. Thus,
$$1 \leq (a + b)y + y(a + b).$$
By Lemma 4 this implies $(a + b)^{-1}$ exists and $0 \leq (a + b)^{-1} \in A_1$. Next observe that
$$a(ya - ay) = (ay)a - a^2y = a(ay) - a^2y = 0$$
and
$$(ya - ay)b = y(ab) - ayb = y(ba) - ayb = (yb)a - ayb = a(ay) - ayb = 0.$$ Put $v = (ya - ay)a$. Then
$$v^2 = (ya - ay)(a(ya - ay))a = 0,$$ and
$$(ya - ay)(a + b) = (ya - ay)a + (ya - ay)b = v + 0 = v.$$ Since $(a + b)^{-1}$ exists, we have $ya - ay = v(a + b)^{-1}$ or $ya = ay + v(a + b)^{-1}$.
Now note, by Lemma 1, we have $y \geq 0$. But from $0 \leq y(a + b) = ya + yb \leq yb + ay + v(a + b)^{-1} \leq 2 + v(a + b)^{-1}$. Thus, $-2 \leq -(a + b)^{-1} \geq 0$ (since $1 \geq a + b \geq 0$).
By Lemma 1, we have $v \geq 0$. But from $0 \leq y(a + b) = ya + yb + v(a + b)^{-1}$, and $(a + b)^{-1} \geq 0$, we get
$$y = (ay + yb)(a + b)^{-1} + v(a + b)^{-2} = c_1 + w,$$ where $0 \leq c_1 = (ay + yb)(a + b)^{-1} \in A_1$, $0 \leq w = v(a + b)^{-2}$.
By Lemma 2, $w^2 = 0$. Finally, observe that $2(a + b) \leq ay + yb$. Since $(a + b)^{-1} \geq 0$, we obtain
$$2 \leq c_1 = (ay + yb)(a + b)^{-1} \in A_1.$$ Now $y = x + 2 = c_1 + w$ or $x = c + w$, where $c = c_1 - 2 \geq 0$. The proof is complete. ∎

Corollary 6. If the dsc-pola $A$ has the MD property and if $u = u_1u_2 = u_2u_1$, where $u_1, u_2, u$ are as in the definition of the decomposition property, then $A = A_1$.

Proof. For any $1 \leq x \in A$, we want to show $x^{-1} \geq 0$. Choose $y \in A$, such that $1 \leq x \leq x + 1 \leq y$. Clearly $2 \leq y$ and $1 \leq y^2 - 1 \leq y^2$. Thus, by the MD property and the assumption, there exists $0 \leq z_1 \leq y$, $0 \leq z_2 \leq y$ such that $y^2 - 1 = z_1z_2 = z_2z_1$ or
$$1 = y(y - z_2) + (y - z_1)z_2 = (y - z_1)y + z_1(y - z_2) = y(y - z_1) + (y - z_2)z_1.$$
Put $0 \leq a = y - z_1$, $0 \leq b = y - z_2$. Then proceed as in Theorem 5. Note now
$1 \geq ay \geq a \geq 0$, $1 \geq ya \geq a \geq 0$, so $ay, ya \in A_1$, hence, $ya - ay \in A_1$. This
implies $v = (ya - ay)a \in A_1$ ($v$ as in the proof of Theorem 5). But $v^2 = 0$;
this by Corollary I. 2.5 of [2] implies $v = 0$. Therefore, $w = v(a + b)^{-2} = 0$;
hence, $2 \leq y = c_1 + w = c_1 \in A_1$. This means $y^{-1} \geq 0$. By Proposition 3 of [3]
we see $x^{-1} \geq 0$, hence, $x \in A_1$, thus, $A = A_1$. □

**Corollary 7.** If $A$ has the MD property, then for any $1 \leq x \in A$, $x^{-1}$
exists and $x^{-1} \leq 1$.

**Proof.** From Theorem 5 we see easily that if $1 \leq x \in A$, then $x = c + w$,
where $1 \leq c \in A_1$, $0 \leq w$, $w^2 = 0$. Since $0 \leq c^{-1} \leq 1$, we have $0 \leq c^{-1}w \leq w$, so
$(c^{-1}w)^2 = 0$. Now $x = c(1 + c^{-1}w)$, thus,
$x^{-1} = (1 - c^{-1}w)c^{-1} = c^{-1} - c^{-1}wc^{-1} \leq c^{-1} \leq 1$. □

**Remark.** The converse of the theorem in general is not true; see the
example at the end.

**Corollary 8.** If $A$ has the MD property, and $w \in A$, $w^2 = 0$, then
$w = w_1 - w_2$, where $0 \leq w_i \in A$, $w_i^2 = 0$ ($i = 1, 2$), and $-v \leq w \leq v$ for some $0 \leq v \in A$, $v^2 = 0$.

**Proof.** Let $w = x_1 - x_2$, $0 \leq x_i$, $i = 1, 2$. By Theorem 5 $x_i = c_i + w_i$,
where $0 \leq c_i \in A_1$, $0 \leq w_i$, $w_i^2 = 0$, so $w = (c_1 - c_2) + (w_1 - w_2)$. Squaring both
sides and using Corollary 7 and Lemma 3 we have

$$w^2 = 0 = (c_1 - c_2)^2 + (c_1 - c_2)(w_1 - w_2) + (w_1 - w_2)(c_1 - c_2)$$
or

$$-(c_1 - c_2)^2 = (c_1 - c_2)(w_1 - w_2) + (w_1 - w_2)(c_1 - c_2).$$

Squaring both sides again and repeatedly using Lemma 2, Lemma 3, and
Corollary 7, we have $(c_1 - c_2)^4 = 0$. But $c_1 - c_2 \in A_1$: by Corollary I. 2.5
of [2] we see easily that $c_1 - c_2 = 0$, so that $w = w_1 - w_2$. By putting $v = w_1 + w_2$, and using Lemma 3, the assertion is now clear. □

**Remark.** By the same method above, we can actually show that for
any $w \in A$, if $w^n = 0$, $n > 2$, then $w^2 = 0$. Furthermore, by this corollary
and Lemma 3, it is quite easy to see that the sum (product) of any two
nilpotents is a nilpotent (zero).

Now the proof of the Main Theorem is straightforward as follows:
For any $x \in A$, $x = x_1 - x_2 = (c_1 + w_1) - (c_2 + w_2) = d + w$, where $0 \leq x_i =
c_i + w_i$, $0 \leq c_i \in A_1$, $0 \leq w_i$, $w_i^2 = 0$ ($i = 1, 2$), and $d = c_1 - c_2 \in A_1$, $w = w_1 - w_2$.
Note that $w^2 = 0$. For the uniqueness part: Suppose $x = d + w = e + u$, $e \in A_1$, $u^2 = 0$. Then $d - e = u - w$. Squaring both sides and using the remark of
Corollary 8 and Corollary I. 2.5 of [2], we see immediately that $d = e$ and
$u = w$. 

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Let \( N = \{ w : w \in A, w^2 = 0 \} \). From Corollary 8 and its remark we know \( N \) is an additive group; it is trivial to verify that \( N \) is a dsc-pola. Now we show that \( N \) has the well-known addition decomposition property:

**Theorem 9.** If \( u_i \in N, u_i \geq 0 \) \((i = 1, 2)\) and \( 0 \leq w \leq u_1 + u_2 \), then there exists \( 0 \leq w_i \leq u_i \) such that \( w = w_1 + w_2 \).

**Proof.** Since \( 0 \leq w \leq (1 + u_1)(1 + u_2) = 1 + u_1 + u_2 \), by the MD property, we have \( w = z_1 z_2 \) where \( 0 \leq z_i \leq 1 + u_i \) \((i = 1, 2)\). By Theorem 5 we obtain easily that \( z_i = a_i + v_i \), where \( 0 \leq a_i \leq 1, 0 \leq v_i \leq u_i \). Now

\[
  w = (a_1 + v_1)(a_2 + v_2) = a_1 a_2 + a_1 v_2 + v_1 a_2.
\]

This implies \( 0 \leq a_1 a_2 \leq w \). Therefore, \( (a_1 a_2)^2 = 0 \). But \( a_1 a_2 \in A_1 \), hence, \( a_1 a_2 = 0 \). Now by putting \( w_1 = v_1 a_2, w_2 = a_1 v_2 \), then the assertion is clear. \( \square \)

**Example 1.** Let \( A \) be the real linear algebra of matrices (real entries) of some given finite order. If \( A \) is partially ordered componentwise, then the diagonal part \( A_1 \) of \( A \) is nothing but all the diagonal matrices. If, in particular, \( A \) consists of the matrices which have the form \( x = [\alpha_{ij}] \) where \( \alpha_{ij} = 0 \) for \( i \neq j \) or \( i \neq 1 \), then the readers are invited to verify that \( A \) has the MD property. Note each element in \( A \) can be written as a diagonal matrix plus a nilpotent matrix.

**Example 2.** \( A\{[x \delta] : x, \delta \text{ are reals}\} \). If we order \( A \) componentwise, then \( A \) is a dsc-pola. It can be verified easily that \( A \) has no MD property, but each element of \( A \) can be decomposed as a diagonal matrix plus a nilpotent matrix. This means the converse of the Main Theorem is not true.

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**References**


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