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ON A MULTIPLICATION DECOMPOSITION THEOREM IN A DEDEKIND σ-COMPLETE PARTIALLY ORDERED LINEAR ALGEBRA

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ABSTRACT. Suppose a Dedekind σ-complete partially ordered linear algebra (dsc-pola) satisfies a certain multiplication decomposition property (see definition below), then we show that this partially ordered linear algebra actually has the same structure of a special class of real matrix algebras, consisting of elements that can be decomposed as diagonal part plus nilpotent part w, such that $w^2=0$.

A *dsc-pola*, denoted by A (or B) is a real linear associative algebra which satisfies the following two conditions: (1) It is partially ordered so that it is a directed partially ordered linear space and $0\leqq xy$ whenever $x, y \in A$, $0\leqq x$, $0\leqq y$. (2) It is Dedekind σ-complete, i.e., if $x_n \in A$, $0\leqq\cdots\leqq x_2\leqq x_1$, then $\inf\{x_n\}$ exists. A dsc-pola A has the Archimedean property: If x, $y \in A$ and $nx\leqq y$ for every positive integer n, then $x\leqq 0$. In this paper we will assume A has a multiplicative identity $1\geqq 0$. Let $I=\{y:y\geqq 1$, and $y^{-1}\geqq 0\}\subset A$. Define $A_1=\bigcup_{y\in I}\{x:-y\leqq x\leqq y\}$. Then it was shown by R. DeMarr that the multiplication of the elements in A_1 is commutative, and A_1 behaves much like an algebra of real-valued functions; moreover, A_1 is a lattice and has no nonzero nilpotent. For the details of the proofs and examples of A_1 we refer to [**2**]. (Note in [**2**], instead of the term dsc-pola, we use *polac*; actually they have the same meaning.) We will call A_1 the *functional* or *diagonal* part of A. Let A be a dsc-pola which has the following multiplication decomposition property (abbreviated as MD):

MD property: If y_1, $y_2 \in A$, $0\leqq y_1$, $0\leqq y_2$, $0\leqq u\leqq y_1y_2$, then there exists $u_i \in A$, $0\leqq u_i\leqq y_i$ $(i=1, 2)$ such that $u=u_1u_2$.

It was shown as Theorem 4 in [**4**] that if A is commutative and has the MD property, then $A=A_1$. In this paper we will drop the commutativity assumption and show the following theorem:

MAIN THEOREM. *If a dsc-pola A has the MD property, then for each $x \in A$, $x=d+v$, where $d \in A_1$, $v^2=0$, and this expression is unique (in the sense that if $x=d+v=e+u$, $e \in A_1$, $u^2=0$, then $d=e$, $v=u$).*

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LEMMA 1. *For any dsc-pola B if $w \in B$, $w^2=0$ and $w \geqq -1$, then $w \geqq 0$.*

PROOF. Since $1+w \geqq 0$, we have $(1+w)^n \geqq 0$ or $1+nw \geqq 0$, for all $n>0$. This means $w \geqq -(1/n)1$ for all n. By the Archimedean property we have $w \geqq 0$. □

LEMMA 2. *For any dsc-pola B if $w \geqq 0$, $w^2=0$, then for any $0 \leqq a \in B_1$ (B_1 is the diagonal part of B), $(aw)^2=(wa)^2=0$.*

PROOF. See the remark of Theorem II. 3.6 of [**2**]. □

LEMMA 3. *If a dsc-pola B has the following property: given any $1 \leqq x \in B$, x^{-1} exists and $x^{-1} \leqq 1$, then for any $0 \leqq w \in B$, $w^n=0$, $n \geqq 2$, we have $w^2=0$; moreover, the sum (product) of positive nilpotents is a nilpotent (zero).*

PROOF. See Theorems II. 3.1, II. 3.2 and its corollary in [**2**]. □

LEMMA 4. *For any dsc-pola B, let $x \in B$, $0 \leqq x \leqq 1$, if there exists $0 \leqq y$ such that $1 \leqq xy+yx$, then x^{-1} exists and $x^{-1} \geqq 1$.*

PROOF. Put $z=1-x \geqq 0$. By assumption we have $1 \leqq xy+yx=(1-z)y+y(1-z)$ or $1 \leqq 1+zy+yz \leqq 2y$. Hence, $2y \geqq 1+z(\frac{1}{2})+(\frac{1}{2})z=1+z$. By induction we will show $2y \geqq 1+\sum_{k=1}^{n} z^k=h_n$ for all n. The assertion is clearly true for $n=1$. If the assertion is true for $n=m$, i.e., $2y \geqq h_m$, then for $n=m+1$, we first observe that $2yz \geqq h_m z$, $2zy \geqq zh_m$ and $h_m z=zh_m$; hence,

$$2y \geqq 1+yz+zy \geqq 1+\tfrac{1}{2}(h_m z+zh_m)=1+zh_m=1+\sum_{k=1}^{m+1} z^k=h_{m+1}.$$

Therefore, h_n is bounded above by $2y$, by Proposition 2 in [**3**] we see

$$1 \leqq h=\sup\{h_n\}=\sum_{0}^{\infty} z^k=(1-z)^{-1} \leqq 2y. \quad \square$$

THEOREM 5. *Let the dsc-pola A have the MD property. If $0 \leqq x \in A$, then $x=c+w$, where $0 \leqq c \in A_1$, $0 \leqq w$ and $w^2=0$.*

PROOF. Put $y=x+2 \geqq 2$. Clearly $1 \leqq y^2-1 \leqq y^2$. By the MD property there exists $z_1, z_2 \in A$ such that $0 \leqq z_1 \leqq y$, $0 \leqq z_2 \leqq y$ and $y^2-1=z_1 z_2$. Thus

$$1=y(y-z_2)+(y-z_1)z_2=(y-z_1)y+z_1(y-z_2).$$

From this we see easily that

$$1 \geqq y(y-z_2) \geqq y-z_2 \geqq 0, \qquad 1 \geqq (y-z_1)y \geqq y-z_1 \geqq 0.$$

Hence, $z_1 \geqq y-1 \geqq 1$, $z_2 \geqq y-1 \geqq 1$. Put $a=y-z_1$, $b=y-z_2$. Then $1 \geqq ay \geqq a \geqq 0$, $1 \geqq yb \geqq b \geqq 0$; this means a, b, ay, yb all belong to A_1; therefore, they

commute with each other. Now $0 \leqq a+b \leqq 1 = az_2 + yb \leqq ay + yb \leqq 2$. Thus,

$$1 \leqq (a+b)y + y(a+b).$$

By Lemma 4 this implies $(a+b)^{-1}$ exists and $0 \leqq (a+b)^{-1} \in A_1$. Next observe that

$$a(ya - ay) = (ay)a - a^2y = a(ay) - a^2y = 0$$

and

$$\begin{aligned}(ya - ay)b &= y(ab) - ayb = y(ba) - ayb = (yb)a - ayb \\ &= a(yb) - ayb = 0.\end{aligned}$$

Put $v = (ya - ay)a$. Then

$$v^2 = (ya - ay)(a(ya - ay))a = 0,$$

and

$$(ya - ay)(a + b) = (ya - ay)a + (ya - ay)b = v + 0 = v.$$

Since $(a+b)^{-1}$ exists, we have $ya - ay = v(a+b)^{-1}$ or $ya = ay + v(a+b)^{-1}$.

Now note, by $1 \geqq yb$, $ay \geqq 0$, we have

$$0 \leqq y(a+b) = ya + yb = yb + ay + v(a+b)^{-1} \leqq 2 + v(a+b)^{-1}.$$

Thus, $-2 \leqq -2(a+b) \leqq v$ (since $1 \geqq a+b \geqq 0$).

By Lemma 1 we have $v \geqq 0$. But from $0 \leqq y(a+b) = yb + ay + v(a+b)^{-1}$, and $(a+b)^{-1} \geqq 0$, we get

$$y = (ay + yb)(a+b)^{-1} + v(a+b)^{-2} = c_1 + w,$$

where $0 \leqq c_1 = (ay+yb)(a+b)^{-1} \in A_1$, $0 \leqq w = v(a+b)^{-2}$.

By Lemma 2, $w^2 = 0$. Finally, observe that $2(a+b) \leqq ay + yb$. Since $(a+b)^{-1} \geqq 0$, we obtain

$$2 \leqq c_1 = (ay + yb)(a+b)^{-1} \in A_1.$$

Now $y = x + 2 = c_1 + w$ or $x = c + w$, where $c = c_1 - 2 \geqq 0$. The proof is complete. □

COROLLARY 6. *If the dsc-pola A has the MD property and if $u = u_1u_2 = u_2u_1$, where u_1, u_2, u are as in the definition of the decomposition property, then $A = A_1$.*

PROOF. For any $1 \leqq x \in A$, we want to show $x^{-1} \geqq 0$. Choose $y \in A$, such that $1 \leqq x \leqq x + 1 \leqq y$. Clearly $2 \leqq y$ and $1 \leqq y^2 - 1 \leqq y^2$. Thus, by the MD property and the assumption, there exists $0 \leqq z_1 \leqq y$, $0 \leqq z_2 \leqq y$ such that $y^2 - 1 = z_1z_2 = z_2z_1$ or

$$\begin{aligned}1 &= y(y - z_2) + (y - z_1)z_2 = (y - z_1)y + z_1(y - z_2) \\ &= y(y - z_1) + (y - z_2)z_1.\end{aligned}$$

Put $0 \leqq a = y - z_1$, $0 \leqq b = y - z_2$. Then proceed as in Theorem 5. Note now $1 \geqq ay \geqq a \geqq 0$, $1 \geqq ya \geqq a \geqq 0$, so ay, $ya \in A_1$, hence, $ya - ay \in A_1$. This implies $v = (ya - ay)a \in A_1$ (v as in the proof of Theorem 5). But $v^2 = 0$; this by Corollary I. 2.5 of [**2**] implies $v = 0$. Therefore, $w = v(a+b)^{-2} = 0$; hence, $2 \leqq y = c_1 + w = c_1 \in A_1$. This means $y^{-1} \geqq 0$. By Proposition 3 of [**3**] we see $x^{-1} \geqq 0$, hence, $x \in A_1$, thus, $A = A_1$. □

COROLLARY 7. *If A has the MD property, then for any $1 \leqq x \in A$, x^{-1} exists and $x^{-1} \leqq 1$.*

PROOF. From Theorem 5 we see easily that if $1 \leqq x \in A$, then $x = c + w$, where $1 \leqq c \in A_1$, $0 \leqq w$, $w^2 = 0$. Since $0 \leqq c^{-1} \leqq 1$, we have $0 \leqq c^{-1}w \leqq w$, so $(c^{-1}w)^2 = 0$. Now $x = c(1 + c^{-1}w)$, thus,

$$x^{-1} = (1 - c^{-1}w)c^{-1} = c^{-1} - c^{-1}wc^{-1} \leqq c^{-1} \leqq 1. \quad \square$$

REMARK. The converse of the theorem in general is not true; see the example at the end.

COROLLARY 8. *If A has the MD property, and $w \in A$, $w^2 = 0$, then $w = w_1 - w_2$, where $0 \leqq w_i \in A$, $w_i^2 = 0$ ($i = 1, 2$), and $-v \leqq w \leqq v$ for some $0 \leqq v \in A$, $v^2 = 0$.*

PROOF. Let $w = x_1 - x_2$, $0 \leqq x_i$, $i = 1, 2$. By Theorem 5 $x_i = c_i + w_i$, where $0 \leqq c_i \in A_1$, $0 \leqq w_i$, $w_i^2 = 0$, so $w = (c_1 - c_2) + (w_1 - w_2)$. Squaring both sides and using Corollary 7 and Lemma 3 we have

$$w^2 = 0 = (c_1 - c_2)^2 + (c_1 - c_2)(w_1 - w_2) + (w_1 - w_2)(c_1 - c_2)$$

or

$$-(c_1 - c_2)^2 = (c_1 - c_2)(w_1 - w_2) + (w_1 - w_2)(c_1 - c_2).$$

Squaring both sides again and repeatedly using Lemma 2, Lemma 3, and Corollary 7, we have $(c_1 - c_2)^4 = 0$. But $c_1 - c_2 \in A_1$; by Corollary I. 2.5 of [**2**] we see easily that $c_1 - c_2 = 0$, so that $w = w_1 - w_2$. By putting $v = w_1 + w_2$, and using Lemma 3, the assertion is now clear. □

REMARK. By the same method above, we can actually show that for any $w \in A$, if $w^n = 0$, $n > 2$, then $w^2 = 0$. Furthermore, by this corollary and Lemma 3, it is quite easy to see that the sum (product) of any two nilpotents is a nilpotent (zero).

Now the proof of the Main Theorem is straightforward as follows: For any $x \in A$, $x = x_1 - x_2 = (c_1 + w_1) - (c_2 + w_2) = d + w$, where $0 \leqq x_i = c_i + w_i$, $0 \leqq c_i \in A_1$, $0 \leqq w_i$, $w_i^2 = 0$ ($i = 1, 2$), and $d = c_1 - c_2 \in A_1$, $w = w_1 - w_2$. Note that $w^2 = 0$. For the uniqueness part: Suppose $x = d + w = e + u$, $e \in A_1 u^2 = 0$. Then $d - e = u - w$. Squaring both sides and using the remark of Corollary 8 and Corollary I. 2.5 of [**2**], we see immediately that $d = e$ and $u = w$.

Let $N=\{w:w \in A,\ w^2=0\}$. From Corollary 8 and its remark we know N is an additive group; it is trivial to verify that N is a dsc-pola. Now we show that N has the well-known addition decomposition property:

THEOREM 9. *If* $u_i \in N$, $u_i \geqq 0$ $(i=1, 2)$ *and* $0 \leqq w \leqq u_1+u_2$, *then there exists* $0 \leqq w_i \leqq u_i$ *such that* $w=w_1+w_2$.

PROOF. Since $0 \leqq w \leqq (1+u_1)(1+u_2)=1+u_1+u_2$, by the MD property, we have $w=z_1z_2$ where $0 \leqq z_i \leqq 1+u_i$ $(i=1, 2)$. By Theorem 5 we obtain easily that $z_i=a_i+v_i$, where $0 \leqq a_i \leqq 1$, $0 \leqq v_i \leqq u_i$. Now

$$w = (a_1 + v_1)(a_2 + v_2) = a_1a_2 + a_1v_2 + v_1a_2.$$

This implies $0 \leqq a_1a_2 \leqq w$. Therefore, $(a_1a_2)^2=0$. But $a_1a_2 \in A_1$, hence, $a_1a_2=0$. Now by putting $w_1=v_1a_2$, $w_2=a_1v_2$, then the assertion is clear. □

EXAMPLE 1. Let A be the real linear algebra of matrices (real entries) of some given finite order. If A is partially ordered componentwise, then the diagonal part A_1 of A is nothing but all the diagonal matrices. If, in particular, A consists of the matrices which have the form $x=[\alpha_{ij}]$ where $\alpha_{ij}=0$ for $i \neq j$ or $i \neq 1$, then the readers are invited to verify that A has the MD property. Note each element in A can be written as a diagonal matrix plus a nilpotent matrix.

EXAMPLE 2. $A\{\left[\begin{smallmatrix}\alpha & \delta\\ 0 & \alpha\end{smallmatrix}\right]:\alpha, \delta$ are reals$\}$. If we order A componentwise, then A is a dsc-pola. It can be verified easily that A has no MD property, but each element of A can be decomposed as a diagonal matrix plus a nilpotent matrix. This means the converse of the Main Theorem is not true.

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