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PROOF OF A POLYNOMIAL CONJECTURE

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ABSTRACT. Let a real polynomial have only real roots, all belonging to an interval I. An inequality is proved, relating the average value of the polynomial between two consecutive roots to its maximal absolute value in I.

In [1] P. Erdös made a conjecture running approximately like this (slightly generalized):

Let $f(x)$ be a polynomial of degree n ($\geqq 2$) all the roots of which are in the interval $[-1, 1]$. The functional

$$F(f) = \int_a^b dx\, |f(x)| \Big/ (b-a) \max_{-1 \leqq x \leqq 1} |f(x)|, \tag{1}$$

where a and b are consecutive roots of $f(x)$, will then assume its maximal value if $f(x)$ is proportional to the Chebyshev polynomial $T_n(cx+d)$, where c and d are suitably chosen constants, and $\{a, b\}$ is an arbitrary pair of roots.

PROOF. Denote by P the set of polynomials (with real coefficients) of degree $n \geqq 2$, all the roots of which are in the interval $[-1, 1]$. Obviously the subset of P consisting of polynomials of fixed norm is compact. Therefore P contains an optimal polynomial f, i.e. $F(f) \geqq F(g)$ for all polynomials $g \in P$. Putting $f_1(x) = f(cx+d)$ we get

$$F(f_1) = \int_a^b dx\, |f(x)| \Big/ (b-a) \max_{-c+d \leqq x \leqq c+d} |f(x)|.$$

Denote the roots of f, ordered according to the magnitude of their indices, by $\{x_j\}$ $(1 \leqq j \leqq n)$. We have $-1 \leqq x_1 \leqq \cdots \leqq x_n \leqq 1$. If, now, $|f(x)|$ did not assume its maximal value between x_1 and x_n, we could set $c = (x_n - x_1)/2$ and $d = (x_n + x_1)/2$, which would give $F(f_1) > F(f)$, a contradiction. But then f_1 must be optimal if f is, and we can assume $-1 = x_1 \leqq \ldots \leqq x_n = 1$.

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For $n=2$ we have $F(f)=\frac{2}{3}$. Assume $n>2$; then f may have multiple roots. Let us investigate the possibility $a=b$. Let $\{f_m;\ 1\leqq m<\infty\}$ be a fundamental sequence of polynomials in P tending towards f. For all f_m we require the smallest root to be -1 and the largest to be 1. Let the roots $b_m\to b$ and $a_m\to a$. If, now, $b=a$, i.e. $b_m-a_m\to 0$, we must have

$$\max_{a_m\leqq x\leqq b_m} |f_m(x)| \Big/ \max_{-1\leqq x\leqq 1} |f_m(x)| \to 0$$

according to the following argument:

We can put $f_m(x)=\prod_{p=1}^{n}(x-x_{p,m})$, where $x_{p,m}\to x_p$ for $m\to\infty$. For $a_m\leqq x\leqq b_m$ we have

$$|f_m(x)| \leqq ((b_m-a_m)/2)^2\cdot 2^{n-2};$$

for m sufficiently high one of the other root intervals remains greater than $2/n$. In this interval we have: $\max|f_m(x)|>(1/n)^n$, which is independent of m. But, according to [2] we have

$$\int_{a_m}^{b_m} dx\,|f_m(x)| < (b_m-a_m)\cdot \max_{a_m\leqq x\leqq b_m} |f_m(x)|\cdot\frac{2}{3},$$

so that $F(f_m)\to 0$ for $m\to\infty$, a contradiction. Therefore, we can assume $-1\leqq a<b\leqq 1$.

Since f is optimal we have $F(f+\varepsilon\phi)\leqq F(f)$ for all polynomials $\phi\in P$, for which $f+\varepsilon\phi\in P$ (ε is a "sufficiently small" positive number). We can assume that $f(x)>0$ for $a<x<b$, and that $\max_{a<x<b} f(x)=f(z)$, where $a<z<b$.

We let ϕ have the same roots as f except for:

(1) ± 1, whose multiplicities are decreased by 1 (note that a subsequent linear transformation of the independent variable (preserving the value of the functional) can restore the condition $-1=x_1\leqq\cdots\leqq x_n=1$);

(2) a (and b), whose multiplicity is increased by 1, if a is a simple root, and decreased by 1, if a is a multiple root;

(3) z, which is a double root for ϕ;

(4) multiple roots c (where $c\neq a$ and $c\neq b$), whose multiplicities are decreased by 2 (double roots for f shall not be roots for ϕ);

(5) two consecutive simple roots c and d satisfying the condition that $|f(x)|$ does not assume its maximal value for $c<x<d$; c and d shall not be roots for ϕ.

Furthermore, $\phi(x)\geqq 0$ for $a<x<b$.

It is seen that it is possible to choose $\phi\in P$ not identically 0 and satisfying these requirements (making $F(f+\varepsilon\phi)>F(f)$ for ε sufficiently small) except in the case where f has the form indicated in the theorem. Q.E.D.

References

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