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EXTREME MEASURABLE SELECTIONS

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ABSTRACT. The extreme points of the set of measurable selections for a set-valued mapping are characterized. As a corollary, the extreme points of the unit ball of the space of "vector-valued L^p functions" are characterized, thus generalizing results of Sundaresan.

1. **Introduction.** Let E be a separable Banach space and $(S, \mathscr{A}, \mu)$ a measure space. A function $f: S \to E$ is called *measurable* if $f^{-1}(B) \in \mathscr{A}$ for each Borel subset B of E. For $1 \leqq p < \infty$, $L_p = L_p(S, \mathscr{A}, \mu; E)$ denotes the Banach space of measurable functions $f: S \to E$ such that

$$\|f\|_p = \left[\int \|f(s)\|^p \, d\mu(s)\right]^{1/p} < \infty.$$

We will always identify functions that are equal almost everywhere.

In [**11**] Sundaresan shows that (with a suitable change in our definition, even for nonseparable E) if $\|f\|_p = 1$ and $f(s)/\|f(s)\| \in \operatorname{ext} U$ for almost all $s \in S$, then f is an extreme point of the unit ball of L_p where $1 < p < \infty$, S is a locally compact Hausdorff space, μ is a regular Borel measure, and ext U is the set of extreme points of the unit ball U of E. In the case where E is a separable conjugate space, Theorem 2 in [**11**] establishes the converse and gives a characterization of the extreme points of the set of measurable functions $f: S \to U$. These results generalize those of [**5**]. Other earlier work for $S = [0, 1]$ and E finite dimensional was done by Karlin [**8**] and Aumann [**1**]. (When the author originally submitted this note, he was unaware of references [**10**] and [**11**]. He thanks the referee for calling them to his attention.)

In Proposition 1 of this note we give a characterization of the extreme points of the set of measurable selections for a set-valued function F. (It was suggested by Aumann in [**1**, p. 11] that this could be done if E were finite dimensional and F had compact convex values.) From this, we obtain four corollaries. Corollaries 2 and 4 strengthen [**11**, Theorem 2]

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and [**1**, Proposition 6.1]. Considerations of this sort arise not only in the applications mentioned in [**1**] but also in control theory (see [**9**]). Corollary 5 provides a generalization of [**11**, Theorem 2] to the case where $(S, \mathscr{A}, \mu)$ is a complete measure space and E is a separable Banach space.

2. **The characterizations.** Let M and N be separable metric spaces and μ a Borel measure on M. By a μ-measurable subset of M, we mean in the usual Carathéodory sense (see [**2**], e.g.). A function $f: M \to N$ is called μ-measurable [resp. Borel measurable] if $f^{-1}(B)$ is μ-measurable [resp. a Borel set] for each Borel set $B \subset N$.

A subset of a metric space is called *analytic* (or *Souslin*) if it is the continuous image of a Borel set in some complete separable metric space. (See [**3**] and [**4**] concerning analytic sets.)

The following theorem was proved by von Neumann [**6**, Lemma 5, p. 448] with N taken as the reals. A careful examination of the proof reveals that it is valid in the more general setting stated below.

THEOREM 1 (VON NEUMANN). *Let M and N be complete separable metric spaces, A an analytic subset of M, and $g: A \to N$ continuous. Let μ be a Borel measure on N. Then $g(A)$ is μ-measurable and there exists a μ-measurable mapping $\phi: g(A) \to M$ such that $g(\phi(x)) = x$ for each $x \in g(A)$.*

Following [**1**] we let 2^S denote the subsets of S. The graph of a mapping $F: T \to 2^S$ is denoted by $\mathscr{G}_F$ and is defined to be $\{(t, s) | s \in F(t)\}$. F is called Borel measurable or analytic according as its graph is. We note, as pointed out in [**1**, p. 2], that a point-valued function is Borel measurable if and only if its graph is. In [**1**, Proposition 2.1] Aumann observed (for $S = [0, 1]$ and μ a Lebesgue measure) the following consequence of Theorem 1 above.

COROLLARY 1. *Let S_1 and S_2 be complete separable metric spaces, $F: S_1 \to 2^{S_2}$ analytic, $F(s) \neq \varnothing$ for each s, and μ a Borel measure on S_1. Then there is a μ-measurable function $f: S_1 \to S_2$ with $f(s) \in F(s)$ for each $s \in S$.*

PROOF. In Theorem 1 above take $M = S_1 \times S_2$, $N = S_1$, $A = \mathscr{G}_F$ and $g(s_1, s_2) = s_1$. The required selection f is the second component of ϕ.

Throughout the remainder of this paper, unless otherwise explicitly stated, E will be a separable Banach space, S a separable complete metric space, and μ a (positive) Borel measure on S.

We denote the complement of a set A by $\tilde{A}$, and $B \sim A$ means $B \cap \tilde{A}$. $B - A$ and $B + A$ are used, for subsets of E, to mean $\{x - y | x \in B, y \in A\}$ and $\{x + y | x \in B, y \in A\}$ respectively.

LEMMA 1. *Let F and G be Borel measurable from S into 2^E. The mappings H_i, $1 \leqq i \leqq 5$, defined below are Borel measurable.*

(1) $H_1 : s \to F(s) \cap G(s)$.

(2) *If* $A \subset S$ *is a Borel set*, $H_2(s) = F(s)$ *for* $s \in A$ *and* $H_2(s) = G(s)$ *for* $s \in \tilde{A}$.

(3) *If* $f : S \to E$ *is Borel measurable and* λ *is a scalar*, $H_3 : s \to f(s) + \lambda F(s)$.

(4) *If* $B \subset E$ *is a Borel set*, $H_4 : s \to B$ *for all* $s \in S$.

(5) $H_5(s) = F(s) \times G(s)$.

PROOF. (1) $\mathscr{G}_F \cap \mathscr{G}_G = \mathscr{G}_{H_1}$.

(2) $\mathscr{G}_{H_2} = [\mathscr{G}_F \cap (A \times E)] \cup [\mathscr{G}_G \cap (\tilde{A} \times E)]$.

(3) Let $\phi : S \times E \to S \times E$ be defined by $\phi(s, x) = (s, \lambda^{-1}(x - f(s)))$ if $\lambda \neq 0$. Then ϕ is Borel measurable and $\phi^{-1}\mathscr{G}_F = \mathscr{G}_{H_3}$. (The case $\lambda = 0$ is trivial.)

(4) $\mathscr{G}_{H_4} = S \times B$.

(5) $\mathscr{G}_{H_5} = \phi^{-1}(\mathscr{G}_F \times \mathscr{G}_G)$ where $\phi(s, x, y) = ((s, x), (s, y))$.

This completes the proof of Lemma 1.

PROPOSITION 1. *Let $F : S \to 2^E$ be Borel measurable with $F(s)$ convex and nonempty for each $s \in S$. Let $F_1(s) = F(s) \sim \operatorname{ext} F(s)$ and suppose that $\{s \mid F_1(s) \neq \varnothing\}$ is a Borel set. $\mathscr{S}_F$ denotes the set of μ-measurable functions $f : S \to E$ such that $f(s) \in F(s)$ for almost all $s \in S$. Then $f \in \operatorname{ext} \mathscr{S}_F$ if and only if $f(s) \in \operatorname{ext} F(s)$ for almost all $s \in S$.*

PROOF. That the condition is sufficient for f to be in $\operatorname{ext} \mathscr{S}_F$ is clear.

Let $A = \{s \mid f(s) \in F_1(s)\}$ and suppose A is not of μ-measure zero. Define $F_2(s) = (F(s) \times F(s)) \cap (E \times E \sim \Delta)$, where Δ is the diagonal of $E \times E$. $\mathscr{G}_{F_2}$ is a Borel set by Lemma 1, and the mapping $(s, x, y) \to (s, 1/2(x+y))$ sends $\mathscr{G}_{F_2}$ continuously onto $\mathscr{G}_{F_1}$. Hence $\mathscr{G}_{F_1}$ is analytic. From [**2**, Propositions 13, 14, p. 97] it follows that there is a Borel measurable function $g : S \to E$ such that $g = f$ a.e. Let $B = \{s \mid g(s) \in F_1(s)\}$. Now, $\mathscr{G}_g \cap \mathscr{G}_{F_1}$ is analytic since each graph is (see [**3**, p. 454 and p. 482]). If π_1 is the canonical projection of $S \times E$ on S, then $\pi_1(\mathscr{G}_g \cap \mathscr{G}_{F_1}) = B$, and therefore B is analytic. S complete and separable implies that B is μ-measurable (see [**4**, Theorem 5.5, p. 50 and Theorem 7.4, p. 52]). Since $g = f$ a.e., the symmetric difference of A and B is of measure zero. It follows that $\mu B > 0$ since otherwise $\mu A = 0$, a contradiction. Now, μ is regular (see [**2**, Corollary 2, p. 347]) so there is a compact set $K \subset B$ with $\mu K > 0$. Let

$$\begin{aligned} G(s) &= [(g(s) - F(s)) \cap (-g(s) + F(s))] \sim \{0\}, && \text{if } s \in K, \\ &= \{0\}, && \text{if } s \notin K. \end{aligned}$$

By Lemma 1, G is Borel measurable and, since $G(s) \neq \varnothing$ for each s, we may apply Corollary 1 to obtain a μ-measurable function $h: S \to E$ such that $h \in \mathscr{S}_G$. This says that h does not vanish on K and that $g+h$ and $g-h$ belong to $\mathscr{S}_F$. Since $g=f$ a.e., it follows that f is not an extreme point of $\mathscr{S}_F$. This completes the proof of the proposition.

The following corollary extends the L^∞ case of Theorem 2 in [**11**].

COROLLARY 2. *If K is a nonempty, convex, Borel subset of E and $\mathscr{S}_K$ is the set of μ-measurable functions $f: S \to K$, then $f \in \operatorname{ext} \mathscr{S}_K$ if and only if $f(s) \in \operatorname{ext} K$ for almost all $s \in S$.*

The next corollary has two consequences, the second of which contains a converse of Theorem 2 in [**5**].

COROLLARY 3. *Let K be a closed, convex, nonempty subset of E and μ a Borel measure on K. If $\mu(K \sim \operatorname{ext} K) > 0$, there is a Borel measurable function $g: K \to E$ such that $g \neq 0$ on a set of positive measure and such that $x \pm g(x) \in K$ for almost all $x \in K$.*

PROOF. First note that $K \sim \operatorname{ext} K$ is analytic (see the proof of Proposition 1) and hence is μ-measurable. Now, letting $S=K$ in Corollary 2, we see that the identity map on K is not an extreme point of $\mathscr{S}_K$. Thus, there is a μ-measurable function $g_0: K \to E$ such that $x \pm g_0(x) \in K$ for almost all $x \in K$ and such that $g_0 \neq 0$ on a set of positive measure. Let g be a Borel measurable function equal μ-a.e. to g_0.

COROLLARY 4. *Let K be a nonempty, closed, convex subset of E and $(S, \mathscr{A}, \nu)$ a measure space complete in the measure theoretic sense. Let $\mathscr{S}$ be the set of functions $f: S \to K$ that are $\mathscr{A}$-measurable; i.e., $f^{-1}(B) \in \mathscr{A}$ for each Borel set $B \subset K$. (We continue to identify functions equal ν-a.e.) Then $f \in \operatorname{ext} \mathscr{S}$ if and only if $f(s) \in \operatorname{ext} K$ for ν-almost all $s \in S$.*

PROOF. Define $\mu B = \nu f^{-1}(B)$ for each Borel set $B \subset K$, and suppose that $A = \{s \mid f(s) \notin \operatorname{ext} K\}$ is not of ν-measure zero. $K \sim \operatorname{ext} K$ is analytic and therefore μ-measurable. If $\mu(K \sim \operatorname{ext} K) = 0$, then by the regularity of μ, there is a Borel set $B \supset K \sim \operatorname{ext} K$ with $\mu B = 0$. Thus, $f^{-1}(B) \supset A$ and $\nu f^{-1}(B) = 0$. By the completeness of $(S, \mathscr{A}, \nu)$, we have $A \in \mathscr{A}$ and $\nu A = 0$, a contradiction. Hence, $\mu(K \sim \operatorname{ext} K) > 0$ and we choose g to be the Borel measurable function guaranteed by Corollary 3. Then $g \circ f \neq 0$ on a set of positive ν-measure and $f(s) \pm g(f(s)) \in K$ for ν-almost all $s \in S$. Hence $f \notin \operatorname{ext} \mathscr{S}$. The converse is clear.

COROLLARY 5. *Let $(S, \mathscr{A}, \nu)$ be as in Corollary 4. Then f is an extreme point of the unit ball of $L_p = L_p(S, \mathscr{A}, \nu; E)$, $1 < p < \infty$, if and only if $\|f\|_p = 1$, and for almost all s in the support S_f of f, $f(s)/\|f(s)\|$ is an extreme point of the unit ball U of E.*

PROOF. Let f be an extreme point of the unit ball of L_p. We apply Corollary 4 to the measure space $(S_f, \mathscr{A}, \nu)$, the convex set U, and the function $h(s)=f(s)/\|f(s)\|$ on S_f. If h is not an extreme point of $\mathscr{S}$, then there exist $h_1, h_2 \in \mathscr{S}$ such that $h_1 \neq h_2$ on a set of positive measure and $h=1/2(h_1+h_2)$. Let $f_j(s)=\|f(s)\|h_j(s)$ for $s \in S_f$ and $f_j(s)=0$ for $s \in S \sim S_f$. It then follows that $f=1/2(f_1+f_2)$, each f_j is in the unit ball of L_p, and $f_1 \neq f_2$ on a set of positive measure. This is a contradiction. Thus, h is an extreme point of $\mathscr{S}$, so by Corollary 4, we have $h(s) \in \text{ext } U$ for almost all $s \in S$.

The proof of the converse may be taken verbatim from [5].

3. **Closing remarks.** The hypothesis of separability and completeness of E and S is necessary to determine that an analytic set is μ-measurable, and that a Borel set in $S \times E$ is in the product sigma-algebra.

If K is a compact convex subset of E then ext K is a $\mathscr{G}_\delta$ set (see [7, Proposition 1.3]). The author does not know whether ext U is a Borel set for arbitrary separable E. As noted earlier, $U \sim \text{ext } U$ is analytic. By [3, Corollary 1, p. 486], therefore, to prove ext U is a Borel set it is enough to prove that it is analytic.

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