PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

# ON FUNCTION SPECTRA

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ABSTRACT. We construct an explicit internal mapping functor for the homotopy category of CW spectra; this, together with our earlier smash product, yields a symmetric monoidal closed category.

**Introduction.** Boardman [2], [9], Adams [1], and the author [5] have defined smash products on the homotopy category of CW spectra which yield a symmetric monoidal category [4]. Further, Boardman and Heller [6] have used Brown's theorem [6, Theorem 12.2] to define an adjoint internal mapping functor, and thus show that such a category is closed. Our construction avoids Brown's theorem.

We thank Alex Heller, William Massey, and John C. Moore for helpful discussions.

1. CW **spectra.** We gave a definition equivalent to the usual definition (e.g., [1], [2]) by applying the Adams completion ([1], [6]) to the following category (in [5]):

DEFINITION 1. A (CW) *prespectrum* $\boldsymbol{X}$ consists of a sequence of pointed CW complexes $\{X_n \mid n \geqq 0\}$, together with cellular inclusions $X_n \wedge S^4 \to X_{n+1}$. Ps is the category of prespectra and strict (continuous, pointed) maps.

Here $S^0 = \partial I$, $S^1 = I/S^0$, $S^n = S^1 \wedge \cdots \wedge S^1$ for $n > 1$, and $S^n = \mathrm{pt}$ otherwise. Denote prespectra with boldface type.

Call a cellular inclusion $\boldsymbol{X}' \subset \boldsymbol{X}$ *cofinal* [1] if the (quadruple) suspensions of each cell of each $X_n$ are eventually in $X'$.

DEFINITION 2. Objects of Ad are prespectra, and

$$\mathrm{Ad}(\boldsymbol{X}, \boldsymbol{Y}) = \operatorname{colim}\{\mathrm{Ps}(\boldsymbol{X}', \boldsymbol{Y}) \mid \boldsymbol{X}' \text{ cofinal in } \boldsymbol{X}\}. \tag{1}$$

For Theorem 5 below we shall need the following equivalent definition of $\mathrm{Ad}(\boldsymbol{X}, \boldsymbol{Y})$ due to Boardman and Heller.

Call a prespectrum $\boldsymbol{X}$ *finite* if, for sufficiently large $n$, $X_n$ is a finite complex and $X_{n+1} = X_n \wedge S^4$. Use (1) to define $\mathrm{Ad}(\boldsymbol{X}, \boldsymbol{Y})$ for finite $\boldsymbol{X}$.

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Received by the editors February 22, 1973.

*AMS (MOS) subject classifications* (1970). Primary 55B20; Secondary 18D15, 55J99.

*Key words and phrases.* Adams category, Boardman category, *h-c* category, generalized cohomology theory, the Brown theorem, realization of singular complex.

Extend this by regarding an arbitrary prespectrum $X$ as the colimit of its finite subspectra.

Define a smash product $\wedge: \mathrm{CW}, \mathrm{Ad} \to \mathrm{Ad}$ by $(K \wedge X)_n = K \wedge X_n$; define homotopy with the cylinder functor $I^* \wedge ?$; and let Ht(Ad) be the resulting homotopy category. See [1] or [5] for details.

We shall also need the category Ws of weak prespectra, which is defined as follows. In Definition 1, replace "CW complex" by "compactly generated space" [8], and replace "cellular inclusion" by "continuous map."

Observe $\mathrm{Ps} \subset \mathrm{Ws}$. Extend Ad( , ) to a functor on $\mathrm{Ad} \times \mathrm{Ws}$.

2. **Construction of the internal mapping functor** MAP. Let $X$ and $Y$ be prespectra. As a first approximation, define a weak prespectrum $\mathrm{Map}(X, Y)$ as follows. Let $S^{-4n}$ be the $-4n$-sphere prespectrum: $(S^{-4n})_i = S^{4i-4n}$; the required inclusions are induced by $S^{4i-4n} \wedge S^4 \cong S^{4i-4n+4}$ for $i+n \geqq 0$. Choose a representative smash product on Ht(Ad) [5, §3]; as in [5], the mapping functor will be independent of this choice. Let $\mathrm{Map}(X, Y)_n = \mathrm{Ad}(S^{-4n} \wedge X, Y)$, with the topology induced from the compactly generated function *spaces* [8] $\mathrm{Map}((S^{-4n} \wedge X)_i, Y_i)$. The maps $S^4 \wedge S^{-4n-4} \to S^{-4n}$ induced the required maps

$$\mathrm{Map}(X, Y)_n \wedge S^4 \to \mathrm{Map}(X, Y)_{n+1}.$$

REMARKS 3. To construct an adjoint to the Boardman-Adams smash product, replace "$S^{-4n}$" by "$S^{-n}$," and "$S^4$" by "$S^1$" in the above construction.

DEFINITION 4. Let $\mathrm{MAP}(X, Y)$ be the (degreewise) realization of the singular complex [7] of the telescope [5] of $\mathrm{Map}(X, Y)$.

Extend MAP to a functor $\mathrm{Ht(Ad)} \times \mathrm{Ht(Ad)} \to \mathrm{Ht(Ad)}$.

3. **The closed structure.** We show that MAP is the required internal mapping functor.

THEOREM 5. $\mathrm{Ht(Ad)}(X \wedge Y, Z) \cong \mathrm{Ht(Ad)}(X, \mathrm{MAP}(Y, Z))$.

PROOF. First observe that, for finite $X$ and $Y$,

$$\mathrm{Ad}(X \wedge Y, Z) \cong \mathrm{Ad}(X, \mathrm{Map}(Y, Z)). \tag{2}$$

By taking colimits, and using the Boardman-Heller completion (§1), we obtain (2), and hence its analogue in Ht(Ad), for arbitrary prespectra. Finally, there are natural weak homotopy equivalences $\mathrm{MAP}(Y, Z) \to \mathrm{Map}(Y, Z)$; thus

$$\mathrm{Ht(Ad)}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Ht(Ad)}(X, \mathrm{MAP}(Y, Z))$$

by [1, Theorem 3.4]. The conclusion follows.

COROLLARY 6. $\pi_0$ MAP( , )$\cong$Ht(Ad)( , ).

This follows from the isomorphism $\pi_0 \cong$Ht(Ad)($S^0$, ), which also shows that Ht(Ad) is normalized [**4**, p. 491].

The remaining coherence conditions for a closed category [**4**, p. 491, Theorem 5.5] are easily verified; their precise statement and proof are omitted.

REMARKS 7. For a weak prespectrum $W$, $\pi_{-*}$ MAP(?, $W$) is a generalized cohomology theory, compare the dual relationship between spectra and generalized homology theories ([**9**], [**1**]). An internal mapping functor on the homotopy category of pointed CW complexes may be constructed analogously with Definition 3; its existence follows from Brown's theorem [**3**].

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