

ON FUNCTION SPECTRA

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ABSTRACT. We construct an explicit internal mapping functor for the homotopy category of CW spectra; this, together with our earlier smash product, yields a symmetric monoidal closed category.

Introduction. Boardman [2], [9], Adams [1], and the author [5] have defined smash products on the homotopy category of CW spectra which yield a symmetric monoidal category [4]. Further, Boardman and Heller [6] have used Brown's theorem [6, Theorem 12.2] to define an adjoint internal mapping functor, and thus show that such a category is closed. Our construction avoids Brown's theorem.

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1. **CW spectra.** We gave a definition equivalent to the usual definition (e.g., [1], [2]) by applying the Adams completion ([1], [6]) to the following category (in [5]):

DEFINITION 1. A (CW) *prespectrum* X consists of a sequence of pointed CW complexes $\{X_n | n \geq 0\}$, together with cellular inclusions $X_n \wedge S^4 \rightarrow X_{n+1}$. $\mathcal{P}s$ is the category of prespectra and strict (continuous, pointed) maps.

Here $S^0 = \partial I$, $S^1 = I/S^0$, $S^n = S^1 \wedge \cdots \wedge S^1$ for $n > 1$, and $S^n = \text{pt}$ otherwise. Denote prespectra with boldface type.

Call a cellular inclusion $X' \subset X$ *cofinal* [1] if the (quadruple) suspensions of each cell of each X_n are eventually in X' .

DEFINITION 2. Objects of Ad are prespectra, and

$$(1) \quad \text{Ad}(X, Y) = \text{colim}\{\mathcal{P}s(X', Y) \mid X' \text{ cofinal in } X\}.$$

For Theorem 5 below we shall need the following equivalent definition of $\text{Ad}(X, Y)$ due to Boardman and Heller.

Call a prespectrum X *finite* if, for sufficiently large n , X_n is a finite complex and $X_{n+1} = X_n \wedge S^4$. Use (1) to define $\text{Ad}(X, Y)$ for finite X .

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Extend this by regarding an arbitrary prespectrum X as the colimit of its finite subspectra.

Define a smash product $\wedge : CW, Ad \rightarrow Ad$ by $(K \wedge X)_n = K \wedge X_n$; define homotopy with the cylinder functor $I^* \wedge ?$; and let $Ht(Ad)$ be the resulting homotopy category. See [1] or [5] for details.

We shall also need the category Ws of weak prespectra, which is defined as follows. In Definition 1, replace “CW complex” by “compactly generated space” [8], and replace “cellular inclusion” by “continuous map.”

Observe $Ps \subset Ws$. Extend $Ad(,)$ to a functor on $Ad \times Ws$.

2. Construction of the internal mapping functor MAP. Let X and Y be prespectra. As a first approximation, define a weak prespectrum $Map(X, Y)$ as follows. Let S^{-4n} be the $-4n$ -sphere prespectrum: $(S^{-4n})_i = S^{4i-4n}$; the required inclusions are induced by $S^{4i-4n} \wedge S^4 \cong S^{4i-4n+4}$ for $i+n \geq 0$. Choose a representative smash product on $Ht(Ad)$ [5, §3]; as in [5], the mapping functor will be independent of this choice. Let $Map(X, Y)_n = Ad(S^{-4n} \wedge X, Y)$, with the topology induced from the compactly generated function spaces [8] $Map((S^{-4n} \wedge X)_i, Y_i)$. The maps $S^4 \wedge S^{-4n-4} \rightarrow S^{-4n}$ induced the required maps

$$Map(X, Y)_n \wedge S^4 \rightarrow Map(X, Y)_{n+1}.$$

REMARKS 3. To construct an adjoint to the Boardman-Adams smash product, replace “ S^{-4n} ” by “ S^{-n} ,” and “ S^4 ” by “ S^1 ” in the above construction.

DEFINITION 4. Let $MAP(X, Y)$ be the (degreewise) realization of the singular complex [7] of the telescope [5] of $Map(X, Y)$.

Extend MAP to a functor $Ht(Ad) \times Ht(Ad) \rightarrow Ht(Ad)$.

3. The closed structure. We show that MAP is the required internal mapping functor.

THEOREM 5. $Ht(Ad)(X \wedge Y, Z) \cong Ht(Ad)(X, MAP(Y, Z))$.

PROOF. First observe that, for finite X and Y ,

$$(2) \quad Ad(X \wedge Y, Z) \cong Ad(X, Map(Y, Z)).$$

By taking colimits, and using the Boardman-Heller completion (§1), we obtain (2), and hence its analogue in $Ht(Ad)$, for arbitrary prespectra. Finally, there are natural weak homotopy equivalences $MAP(Y, Z) \rightarrow Map(Y, Z)$; thus

$$Ht(Ad)(X, Map(Y, Z)) \cong Ht(Ad)(X, MAP(Y, Z))$$

by [1, Theorem 3.4]. The conclusion follows.

COROLLARY 6. $\pi_0 \text{MAP}(,) \cong \text{Ht}(\text{Ad})(,)$.

This follows from the isomorphism $\pi_0 \cong \text{Ht}(\text{Ad})(S^0,)$, which also shows that $\text{Ht}(\text{Ad})$ is normalized [4, p. 491].

The remaining coherence conditions for a closed category [4, p. 491, Theorem 5.5] are easily verified; their precise statement and proof are omitted.

REMARKS 7. For a weak prespectrum W , $\pi_{-*} \text{MAP}(?, W)$ is a generalized cohomology theory, compare the dual relationship between spectra and generalized homology theories ([9], [1]). An internal mapping functor on the homotopy category of pointed CW complexes may be constructed analogously with Definition 3; its existence follows from Brown's theorem [3].

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