KAHANE'S CONSTRUCTION AND THE WEAK SERIAL COMPLETENESS OF $L^1$

E. A. HEARD

Abstract. If $\lambda$ is a normalized Lebesgue measure (arc length/2$\pi$) on $T=\{|z|=1\}$, the Gelfand map permits $L^\infty(\lambda)^*$ to be identified with $M(X)$, the space of finite Baire measures on $X$, the maximal ideal space of $L^\infty(\lambda)$. The measure $m_0$ in $M(X)$ represents $\lambda$: $\int_X f \, dm_0 = \int_T f \, d\lambda$ for all $f \in L^\infty(\lambda)$. Furthermore $\mu \ll m_0$ if and only if $\mu$ represents some measure of the form $\phi \, d\lambda$, $\phi \in L^1(\lambda)$. Using this fact and a sum constructed by J. P. Kahane, $\sum \hat{h}^n(\cdot)$ when $\hat{h}$ is an appropriate function guaranteed by Urysohn's lemma, develops a proof that $L^1(\lambda)$ is weakly sequentially complete.

1. Introduction. If $T=\{|z|=1\}$ and $\lambda$ is a normalized Lebesgue (arc length/2$\pi$) measure on $T$, then the Gelfand maps and the Riesz representation theorems permit the isometric and isomorphic identification of $L^\infty(\lambda)^*$ (the dual space of $L^\infty$) with $M(X)$ the space of finite Baire measures on the maximal ideal space $X$ of the Banach algebra $L^\infty$. The measure $m_0$ in $M(X)$ represents the linear functional

$$L(f) = \int_T f \, d\lambda = \int_X f \, dm_0, \quad f \in L^\infty.$$  

It is natural to show that $L^1(\lambda)$ and $L^1(m_0)$ are the same space in different settings. They are twins in the following sense:

**Theorem 1.** If $\mu \in M(X)$ then $\mu \ll m_0$ if, and only if, there is a function $\phi$ in $L^1(\lambda)$ such that

$$\int_X f \, d\mu = \int_T f\phi \, d\lambda, \quad f \in L^\infty.$$

In this case $\|\mu\|_{M(X)} = \|\phi\|_{L^1(\lambda)}$.

The author's initial proof of Theorem 1 used the fact that $L^1(\lambda)$ is weakly sequentially complete. However, Theorem 1 may be proved independently of this well-known theorem and used with a technique of J. P. Kahane to, in fact, prove it, in turn.
Theorem 2. (L is weakly sequentially complete.) If \( \{\phi_n \in L^1(\lambda) \mid n = 1, 2, \cdots \} \) and \( \lim_{n \to \infty} \int f \phi_n \, d\lambda \) exists for each \( f \) in \( L^\infty(\lambda) \), then there is a \( \phi \in L^1(\lambda) \) with

\[
\lim_n \int f \phi_n \, d\lambda = \int f \phi \, d\lambda.
\]

2. Main results. First let us recall some things about the Gelfand maps which identify

\[ L^\infty(\lambda) \rightarrow C(X) \]

and

\[ L^\infty(\lambda)^* \rightarrow C(X)^* \cong M(X), \]

when \( X \) is the maximal ideal space of \( L^\infty(\lambda) \). The first map, \( f \rightarrow \hat{f} \), is an isometric ring isomorphism which preserves complex conjugation [2, p. 170]. The dual map in (3), \( L \rightarrow L^\ast \), is defined by \( L(\hat{g}) = \hat{L}(g) \), \( g \in L^\infty \). The Riesz representation theorem identifies \( C(X)^* \) with \( M(X) \). The measure \( m_0 \) in \( M(X) \) represents \( \lambda \) as in (1):

\[
\int_X f \, dm_0 = \int_T f \, d\lambda, \quad f \in L^\infty.
\]

Theorem 1. A linear functional \( F \) in \( L^\infty(\lambda)^* \) is represented by a measure \( \mu \) in \( M(X) \) which is absolutely continuous with respect to \( m_0 \) if and only if there is a function \( \phi \) in \( L^1(\lambda) \) with

\[
F(g) = \int_X \hat{g} \, d\mu = \int_T g \phi \, d\lambda, \quad g \in L^\infty.
\]

In this case \( \| \mu \|_X = \| \phi \|_1 \).

Proof. First suppose \( \mu \) is a measure in \( M(X) \) with the Lebesgue decomposition \( d\mu = C_E \, dm_0 \) where \( C_E \) is the characteristic function of a closed \( G_\delta \) set \( E \subset X \). The sets of the form \( U = \{ x \in X \mid \hat{C}_K(x) = 1, K \subset T \text{ is } \lambda\text{-measurable} \} \) form a basis for the topology of \( X \) [2, pp. 169-170]. This together with Urysohn's lemma insures the existence of a sequence \( \hat{u}_n \in C(X) \) with \( \lim \hat{u}_n = C_E \). By (1) and the bounded convergence theorem for all \( h \) in \( L^\infty \),

\[
\lim \int_T h u_n \, d\lambda = \lim \int_X \hat{h} \hat{u}_n \, dm_0 = \int_X \hat{h} \, d\mu.
\]

Since \( f \rightarrow \hat{f} \) is a ring and lattice isomorphism

\[
\int_T |u_n - u_m|^2 \, d\lambda = \int_X |\hat{u}_n - \hat{u}_m|^2 \, dm_0.
\]
The completeness of $L^2(\lambda)$ lets us choose a subsequence and a $\phi$ in $L^2$ with $\lim u_n = \phi$ a.e.-$\lambda$. Now the bounded convergence theorem shows that the left-hand side of (4) converges.

(5) $$\int_T h \phi \, d\lambda = \int_X h \, d\mu, \quad h \in L^\infty.$$ 

Consider the set

$$M = \text{span}\{C_E \, dm_0 \mid E \text{ (closed } G_\delta \text{) } \subset X\} \subset M(X).$$

If $j$ is the natural embedding of $L^1(\lambda)$ into its second dual $L^\infty(\lambda)^*$ [1, p. 66], we have shown that the image of $j(L^1(\lambda))$ under the dual Gelfand map (3) contains $M$. Hence it contains $L^1(m_0)$ since $j$ is another isometric isomorphism and $M$ is a dense subset of $L^1(m_0)$ (the latter to be thought of as those measures in $M(X)$ absolutely continuous with respect to $m_0$).

(Observe at this point that if the weak sequential completeness of $L^1(\lambda)$ is assumed true, (4) allows us to assert directly that $\phi \in L^1(\lambda)$ exists with (5) holding.)

Since, by (5),

$$\left| \int_T f \phi \, d\lambda \right| = \left| \int_X \hat{f} \, d\mu \right|, \quad f \in L^\infty(\lambda),$$

taking the supremum over those $f$ with $\|f\|_\infty \leq 1$ shows $\|\phi\|_1 = \|\mu\|_X$.

Conversely let $\phi \in L^1(\lambda)$ and $\mu \in M(X)$ represent $F_\phi$:

(6) $$F_\phi(g) = \int_T g \phi \, d\lambda = \int_X \hat{g} \, d\mu, \quad g \in L^\infty.$$ 

Let $E \subset X$ be a closed $G_\delta$ set with $m_0(E) = 0$. We shall show $\mu(E) = 0$, and thus $\mu \ll m_0$.

There is a sequence $\{\hat{u}_n \in C(X) \mid n = 1, 2, \ldots\}$ with $0 \leq \hat{u}_n \leq 1$ and $\lim u_n = C_E$. Since $\hat{u}_n \geq 0$, $u_n \geq 0$ a.e.-$\lambda$. From (1),

$$0 = m_0(E) = \lim \int_X \hat{u}_n \, dm_0 = \lim \int_T |u_n| \, d\lambda.$$ 

Thus $u_n$ converges to zero in $L^1(\lambda)$. Taking a subsequence suppose $u_n$ converges to zero a.e.-$\lambda$.

By the Lebesgue dominated convergence theorem and (6) we see

$$0 = \lim \int_T u_n \phi \, d\lambda = \lim \int_X \hat{u}_n \, d\mu = \mu(E).$$

This ends the proof.

**Theorem 2.** $L^1(\lambda)$ is weakly sequentially complete.
PROOF. Let \( \{\phi_n \in L^1(\lambda) | n = 1, 2, \cdots \} \) have the property that

\[
L(f) = \lim_{n} \int_T f\phi_n \, d\lambda
\]

exists for each \( f \) in \( L^\infty(\lambda) \). Let \( \mu \) be that measure in \( M(X) \) which represents \( L \); i.e. \( \int_X f \, d\mu = L(f) \). That such a measure \( \mu \) exists and is unique is a consequence of the principle of uniform boundedness and our identification of \( L^\infty(\lambda) \) and \( M(X) \).

If we show that \( \mu \) is absolutely continuous with respect to \( m_0 \) then Theorem 1 says there is a \( \phi \) in \( L^1(\lambda) \) such that

\[
L(f) = \lim_{n} \int_T f\phi_n \, d\lambda = \int_T f\phi \, d\lambda
\]

and we are done.

Let \( d\mu = g \, dm_0 + d\beta \) be the Lebesgue decomposition of \( \mu \) with respect to \( m_0 \) with \( \beta \perp m_0 \). Suppose \( \beta \neq 0 \); i.e. that there is a closed \( G_\delta \) set \( E \subset X \) with \( m_0(E) = 0 \) and \( \int_E d\beta = \int_E d\mu = a \neq 0 \).

By a standard application of Urysohn’s lemma there is a function \( h \) in \( C(X) \) with \( 0 \leq h \leq 1 \), \( h(E) = 1 \) and \( h < 1 \) off \( E \). Taking powers of \( h \), we may construct a sequence \( \{h_n = h^{k(n)} | n = 1, 2, \cdots \} \) so that for every strictly increasing sequence \( \{n(j) | j = 1, 2, \cdots \} \) of positive integers the series

\[
s = \sum_{j=1}^{\infty} (-1)^j h_n(j)
\]

has uniformly bounded partial sums and converges uniformly on compact subsets of \( X \setminus E \) [3].

Now Kahane’s construction requires:

(a) \( \lim_{n} \left| \int h_n \, d\mu \right| = a \neq 0 \).

(b) \( \lim_{j} \int h_n(\phi_j) \, d\lambda = \lim_{j} \int h_n\phi_j \, dm_0 = \int_X h_n \, d\mu, \quad n = 1, 2, \cdots \).  

(c) \( \lim_{n} \int_X h_n\phi_j \, dm_0 = 0, \quad j = 1, 2, \cdots \).

(d) There is a function \( f \) in \( L^\infty \) with \( f = s \) a.e.-\( m_0 \) and hence \( \lim_{j \to \infty} \int_X s\phi_j \, dm_0 \) exists.

In (b), (c), and (d) the functions \( \phi_j \in L^1(m_0) \) are guaranteed by Theorem 1. To see (d), observe that the sequence of partial sums

\[
\left\{ s_n = \sum_{p=1}^{n} (-1)^p h_n(p) \right\}
\]
converges to \( s \) in \( L^2(m_0) \). The complex conjugate preserving Gelfand isomorphism and (1) yield

\[
\int_X |\hat{s}_n - \hat{s}_m|^2 \, dm_0 = \int_T |s_n - s_m|^2 \, d\lambda.
\]

Thus \( \{s_n\}_{n=1, 2, \cdots} \) is uniformly bounded and a Cauchy sequence in \( L^2(\lambda) \). However, if \( s_n \to f \) in \( L^2(\lambda) \) then \( s = f \) a.e.-\( m_0 \) since

\[
\lim_{n \to \infty} \int_X |s_n - f|^2 \, d\lambda = \lim_{n \to \infty} \int_X |\hat{s}_n - \hat{f}|^2 \, dm_0 = \lim_{n \to \infty} \int_X |s_n - s|^2 \, dm_0 = 0.
\]

The rest of the argument parallels that of Kahane in [3] to show \( \beta = 0 \).

3. Comments. It is hardly surprising that Kahane's methods yield the weak sequential completeness of \( L^1 \) since his work was an effort to decide the question: "Is \( L^1/H^1 \) weakly sequentially complete?" This question has since been answered affirmatively by Michael C. Mooney [4].

The series forming \( s \) in (7) is a useful construction. A modification of this series is used in [5] to develop the existence of certain interpolation sets.

Theorem 1 extends to the case of an arbitrary \( \sigma \)-finite measure \( \lambda \); the arguments in [2] are valid without requiring the special properties of Lebesgue measure or \( \{|z|=1\} \). However, Theorem 1 is dependent on the fact that \( L^\infty = (L^1)^* \) and this identification may fail for a non \( \sigma \)-finite measure [6].

A standard proof of Theorem 2 reduces the case of an arbitrary measure to the case of a \( \sigma \)-finite measure [1, pp. 289–291].

REFERENCES


DEPARTMENT OF MATHEMATICS AND PHYSICS, KENTUCKY STATE UNIVERSITY, FRANKFORT, KENTUCKY 40601