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LOCALLY MODULAR LATTICES AND LOCALLY DISTRIBUTIVE LATTICES

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ABSTRACT. A locally modular (resp. locally distributive) lattice is a lattice with a congruence relation and each of whose equivalence class has sufficiently many elements and is a modular (resp. distributive) sublattice. Both the lattice of all closed subspaces of a locally convex space and the lattice of projections of a locally finite von Neumann algebra are locally modular. The lattice of all T_1-topologies of an infinite set is locally distributive.

Introduction. In this paper, a lattice L is called locally modular (resp. locally distributive) when L has a congruence relation θ such that each equivalence class by θ which contains sufficiently many elements is a modular (resp. distributive) sublattice. Any locally distributive lattice is locally modular evidently, and it is shown in §1 that any locally modular lattice is both upper and lower semimodular in the sense of Birkhoff **[2]**. Moreover in this section it is proved that both the lattice of all closed subspaces of a locally convex space and the lattice of all projections of a locally finite von Neumann algebra are locally modular.

It was proved by Larson and Thron **[5]** that the lattice of all T_1-topologies on an infinite set is both upper and lower semimodular. Generalizing this result, it is shown in §2 that the lattice of all T_1-topologies is locally distributive. Moreover, the final theorem of **[5]** is formulated as a theorem on locally distributive lattices.

In the last section, we determine the form of standard elements in the dual of the lattice of T_1-topologies. This result shows us that this lattice has infinitely many standard elements but has no neutral elements except 0 and 1.

1. **Locally modular lattices.** An equivalence relation θ in a lattice L is called a *congruence relation* when it satisfies the following condition:

$$\text{If } a_1 \equiv b_1\ (\theta) \text{ and } a_2 \equiv b_2\ (\theta)$$
$$\text{then } a_1 \vee a_2 \equiv b_1 \vee b_2\ (\theta) \text{ and } a_1 \wedge a_2 \equiv b_1 \wedge b_2\ (\theta).$$

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Then, for any $a \in L$, the equivalence class $[a]=\{x \in L; x\equiv a\ (\theta)\}$ is a sublattice of L. Moreover, if $x, y \in [a]$ and $x<y$ then the interval $L[x, y]=\{z \in L; x\leqq z\leqq y\}$ is contained in $[a]$ (see [**2**, p. 27]).

In a lattice, we write $a<\cdot b$ when b covers a.

DEFINITION. A lattice L is called *locally modular* when there exists a congruence relation θ in L satisfying the following three conditions:

(θ_1) If $a\neq 1$ in L then there exists $b \in L$ such that $b>a$ and $b\equiv a\ (\theta)$, and if $a\neq 0$ then there exists $b \in L$ such that $b<a$ and $b\equiv a\ (\theta)$.

(θ_2) If $a<\cdot b$ then $a\equiv b\ (\theta)$.

(θ_M) For any $a \in L$, the sublattice $[a]$ is modular.

L is called *locally distributive* when, in the above definition, (θ_M) is replaced by the following condition:

(θ_D) For any $a \in L$, the sublattice $[a]$ is distributive.

Evidently, any locally distributive lattice is locally modular. The two conditions (θ_1) and (θ_2) assert that each sublattice $[a]$ contains sufficiently many elements.

THEOREM 1.1. *Any locally modular lattice L is both upper and lower semimodular in the sense of Birkhoff* [**2**].

PROOF. Let $a\wedge b<\cdot a$ and $a\wedge b<\cdot b$ in L. Then we have $a\equiv b\ (\theta)$ by (θ_2), and hence $(a, b)M^*$ and $(b, a)M^*$ by (θ_M) (see (1.7) of [**6**]). Hence we have $b<\cdot a\vee b$ and $a<\cdot a\vee b$ by (7.5.4) of [**6**]. Thus L is upper semimodular. Similarly we can prove that L is lower semimodular.

A lattice L with 0 and 1 is called a *DAC-lattice* when both L and its dual L^* are atomistic lattices with the covering property (see [**6**, §27]). We shall prove that any DAC-lattice is locally modular. We write $\mathscr{F}(L)$ for the set of all finite elements and write $\Omega(L)$ for the set of all atoms of L.

LEMMA 1.1. *Let a and b be elements of a DAC-lattice L.*

(i) *There exists $u \in \mathscr{F}(L)$ such that $a\vee u=b$ if and only if there exists $u^* \in \mathscr{F}(L^*)$ such that $b\wedge u^*=a$.*

(ii) *There exists $u \in \mathscr{F}(L)$ such that $a\vee u=b\vee u$ if and only if there exists $u^* \in \mathscr{F}(L^*)$ such that $a\wedge u^*=b\wedge u^*$.*

PROOF. (i) If $a\vee u=b$ with $u\in\mathscr{F}(L)$, then by the covering property there exists a connected chain $a=x_0<\cdot x_1<\cdot\cdots<\cdot x_n=b$. Since L is dual-atomistic, there exist dual atoms h_i $(i=1,\cdots,n)$ such that $h_i\geqq x_{i-1}$ and $h_i\ngeqq x_i$. Putting $u^*=h_1\wedge\cdots\wedge h_n$, we have $u^* \in \mathscr{F}(L^*)$ and $b\wedge u^*=a$. The converse statement can be proved similarly. Moreover, it is easily seen that the statement (ii) follows from (i).

THEOREM 1.2. *Let L be a DAC-lattice. L is locally modular if we define $a\equiv b\ (\theta)$ by $a\vee u=b\vee u$ for some $u \in \mathscr{F}(L)$.*

PROOF. It is evident that θ is an equivalence relation. θ is a congruence relation by Lemma 1.1(ii), and it satisfies (θ_1) since L is atomistic and dual-atomistic. It satisfies (θ_2) evidently. When $a\equiv b\ (\theta)$, there exists $u^* \in \mathscr{F}(L^*)$ with $a\wedge u^*=b\wedge u^*=c$. It follows from Lemma 1.1 that there exist $u, v \in \mathscr{F}(L)$ such that $c\vee u=a$, $c\vee v=b$. Since L is finite-modular by (27.6) of **[6]**, we have $(a, b)M^*$ by (27.12) of **[6]**. Hence, θ satisfies (θ_M).

By (31.10) of **[6]**, this theorem implies the following result.

COROLLARY. *The lattice of all closed subspaces of a locally convex space is locally modular.*

Next, let L be a relatively complemented lattice with 0 and 1. The following condition is considered in §35 of **[6]**:

(J) L has a join-dense p-ideal J whose elements are all modular.

It follows from (35.6) of **[6]** that L^* also satisfies (J) by using $J^*=\{x \in L;\ x \text{ has a complement } x' \in J\}$ instead of J. An important example of such a lattice is a locally finite dimension lattice defined in (35.15) of **[6]**.

LEMMA 1.2. *Let a and b be elements of a relatively complemented lattice L, with* 0 *and* 1, *satisfying* (J).

(i) *There exists $u \in J$ such that $a\vee u=b$ if and only if there exists $u^* \in J^*$ such that $b\wedge u^*=a$.*

(ii) *There exists $u \in J$ such that $a\vee u=b\vee u$ if and only if there exists $u^* \in J^*$ such that $a\wedge u^*=b\wedge u^*$.*

PROOF. If $a\vee u=b$ with $u \in J$, then taking a complement u^* of b in the interval $L[a, 1]$, we have $b\wedge u^*=a$ and $u\vee u^*=u\vee a\vee u^*=b\vee u^*=1$. Hence, $u^* \in J^*$ by the statement (1) in the proof of (35.6) of **[6]**. The converse statement can be proved similarly. The statement (ii) follows from (i).

THEOREM 1.3. *Let L be a relatively complemented lattice, with* 0 *and* 1, *satisfying* (J). *L is locally modular if we define $a\equiv b\ (\theta)$ by $a\vee u=b\vee u$ for some $u \in J$.*

PROOF. It follows from Lemma 1.2 that θ is a congruence relation. θ satisfies (θ_1) since J (resp. J^*) is join-dense in L (resp. L^*). It satisfies (θ_2) evidently. When $a\equiv b\ (\theta)$, we can prove $(a, b)M^*$ by the same way as in the proof of Theorem 1.2, using (35.10) of **[6]** instead of (27.12).

COROLLARY. *Any locally finite dimension lattice is locally modular. Especially, the lattice of all projections of a locally finite AW^*-algebra is locally modular* (*see* (37.16) *of* **[6]**).

2. Locally distributive lattices.

LEMMA 2.1. *Let L be an atomistic lattice. A congruence relation θ in L satisfies the condition (θ_D) if it satisfies the following condition:*

(Ω_D) *If $a \equiv b$ (θ) in L and if p is an atom of L such that $p \leqq a \vee b$ then either $p \leqq a$ or $p \leqq b$.*

PROOF. For $x, y, z \in [a]$, we have $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$, since if p is an atom with $p \leqq (x \vee y) \wedge z$ then $p \leqq (x \wedge z) \vee (y \wedge z)$ by (Ω_D). Similarly, $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ holds.

LEMMA 2.2. *Let L be a locally distributive atomistic lattice whose congruence relation θ satisfies (Ω_D). If $x < \cdot a$, $y < \cdot a$ in L and if there exists an atom p of L such that $a = x \vee p = y \vee p$ then $x = y$.*

PROOF. Evidently $p \nleqq x$ and $p \nleqq y$. Since $x \equiv a \equiv y$ (θ) by (θ_2), we have $p \nleqq x \vee y$ by (Ω_D), and hence $x \vee y < a$. Since $x < \cdot a$ we have $x = x \vee y$, and similarly $y = x \vee y$.

THEOREM 2.1. *Let L be a complete locally distributive atomistic lattice whose congruence relation θ satisfies (Ω_D). For any $a \in L$, we put $\Gamma(a) = \{x \in L;\ x < \cdot a\}$. If we put $a(M) = \bigwedge (x;\ x \in M)$ for every subset M of $\Gamma(a)$ $(a(\varnothing) = a)$, then the set $\{a(M);\ M \subset \Gamma(a)\}$ is a complete sublattice of L which is dual isomorphic to the Boolean lattice formed by all subsets of $\Gamma(a)$.*

PROOF. Let $\{M_\alpha;\ \alpha \in I\}$ be an arbitrary family of subsets of $\Gamma(a)$. The equation $a(\bigcup_\alpha M_\alpha) = \bigwedge_\alpha a(M_\alpha)$ holds evidently and we shall prove $a(\bigcap_\alpha M_\alpha) = \bigvee_\alpha a(M_\alpha)$ (we denote by $\cup$ and $\cap$ the union and the intersection respectively). It suffices to show that if p is an atom with $p \leqq a$ and $p \nleqq \bigvee_\alpha a(M_\alpha)$ then $p \nleqq a(\bigcap_\alpha M_\alpha)$. For every α, there exists $x_\alpha \in M_\alpha$ with $p \nleqq x_\alpha$, since $p \nleqq a(M_\alpha)$. Then, since $x_\alpha \vee p = a$, it follows from Lemma 2.2 that $x_\alpha = x_\beta$ for every $\alpha, \beta \in I$. Hence, $p \nleqq a(\bigcap_\alpha M_\alpha)$. Therefore, $\{a(M);\ M \subset \Gamma(a)\}$ is a complete sublattice of L. Moreover, it is easy to prove by Lemma 2.2 that the mapping $M \rightarrow a(M)$ is one-to-one. This completes the proof.

Next, we shall give an example of a locally distributive lattice whose congruence relation satisfies (Ω_D). Let X be an infinite set. A topology on X is denoted by the collection $\mathscr{T}$ of all open sets. $\mathscr{T}$ is a T_1-topology if and only if $\mathscr{T}$ contains all cofinite subsets of X. The set $L_T(X)$ of all T_1-topologies on X forms a complete lattice, ordered by set inclusion, that is, $\mathscr{T}_1 < \mathscr{T}_2$ means that $\mathscr{T}_2$ is finer than $\mathscr{T}_1$. The greatest element of $L_T(X)$ is the discrete topology and the least element is the cofinite topology (see [7, §1]).

For any subset Y of X, we denote by $\mathscr{P}(Y)$ the collection of all subsets of Y. It was shown in [3] and [7] that a dual-atom of $L_T(X)$, which is called a nonprincipal ultratopology, has the form

$$\mathscr{T}(x, \mathscr{U}) = \mathscr{P}(X - \{x\}) \cup \mathscr{U}$$

where $x \in X$ and $\mathscr{U}$ is a nonprincipal ultrafilter on X, and it follows from Theorem 1.1 of [7] that $L_T(X)$ is dual-atomistic. We remark that $L_T(X)$ is not atomistic.

THEOREM 2.2. *Let X be an infinite set. The lattice $L_T(X)$ of T_1-topologies on X is locally distributive if we define $\mathscr{T}_1 \equiv \mathscr{T}_2$ (θ) by $\mathscr{T}_1 \cap \mathscr{P}(X-F) = \mathscr{T}_2 \cap \mathscr{P}(X-F)$ for some finite subset F of X (i.e. $\mathscr{T}_1$ coincides with $\mathscr{T}_2$ on some cofinite subset). Moreover, this congruence relation θ satisfies (Ω_D) in the dual of $L_T(X)$.*

PROOF. It is easy to verify that θ is a congruence relation. Let $\mathscr{T} \in L_T(X)$. If $\mathscr{T}$ is not discrete, then there exists $x \in X$ such that $\{x\} \notin \mathscr{T}$. Putting $\mathscr{T}_1 = \mathscr{T} \cup \{G \cup \{x\}; G \in \mathscr{T}\}$, we have $\mathscr{T} < \mathscr{T}_1 \in L_T(X)$ and $\mathscr{T}_1 \equiv \mathscr{T}$ (θ). If $\mathscr{T}$ is not the cofinite topology, then there exists a dual-atom $\mathscr{T}(x, \mathscr{U})$ such that $\mathscr{T}(x, \mathscr{U}) \ngeq \mathscr{T}$. Putting $\mathscr{T}_2 = \mathscr{T} \wedge \mathscr{T}(x, \mathscr{U})$, we have $\mathscr{T}_2 < \mathscr{T}$ and $\mathscr{T}_2 \equiv \mathscr{T}$ (θ). Hence, θ satisfies (θ_1). If $\mathscr{T}_1 \prec \mathscr{T}_2$, then there exists a dual-atom $\mathscr{T}(x, \mathscr{U})$ such that $\mathscr{T}_1 = \mathscr{T}_2 \wedge \mathscr{T}(x, \mathscr{U})$. Hence, θ satisfies (θ_2).

Next, we shall show that θ satisfies (Ω_D) in the dual of $L_T(X)$, that is, if $\mathscr{T}_1 \cap \mathscr{P}(X-F) = \mathscr{T}_2 \cap \mathscr{P}(X-F)$ and $\mathscr{T}(x, \mathscr{U}) \geqq \mathscr{T}_1 \wedge \mathscr{T}_2$ then $\mathscr{T}(x, \mathscr{U}) \geqq \mathscr{T}_1$ or $\mathscr{T}_2$. If we had $\mathscr{T}(x, \mathscr{U}) \ngeqq \mathscr{T}_i$ for $i=1, 2$, then there would exist $G_i \in \mathscr{T}_i$ such that $G_i \notin \mathscr{T}(x, \mathscr{U})$. Since $G_i \notin \mathscr{P}(X-\{x\}) \cup \mathscr{U}$, we have $x \in G_i \notin \mathscr{U}$, and then $G_1 \cup G_2 \notin \mathscr{U}$ since $\mathscr{U}$ is an ultrafilter. We put

$$G = (G_1 \cup G_2) \cap \{G_1 \cup (X - F)\} \cap \{G_2 \cup (X - F)\}.$$

Since $G_2 - F \in \mathscr{T}_2 \cap \mathscr{P}(X-F) \subset \mathscr{T}_1$, we have $(G_1 \cup G_2) \cap \{G_1 \cup (X-F)\} = G_1 \cup (G_2 - F) \in \mathscr{T}_1$. Moreover, $G_2 \cup (X-F) \in \mathscr{T}_1$ since it is a cofinite subset. Hence, we have $G \in \mathscr{T}_1$, and similarly $G \in \mathscr{T}_2$. On the other hand, since $x \in G$ and $G \subset G_1 \cup G_2 \notin \mathscr{U}$, we have $G \notin \mathscr{T}(x, \mathscr{U})$. This contradicts that $\mathscr{T}(x, \mathscr{U}) \geqq \mathscr{T}_1 \wedge \mathscr{T}_2$.

In the dual of $L_T(X)$, since θ is a congruence relation satisfying (Ω_D), θ satisfies (θ_D) by Lemma 2.1. Hence, θ satisfies (θ_D) in $L_T(X)$ also. Therefore $L_T(X)$ is locally distributive.

REMARK. It follows from Theorem 1.1 that the above theorem is a generalization of [**5**, Theorems 3 and 4]. Moreover, Lemma 2.2 and Theorem 2.1 are lattice theoretical generalizations of [**5**, Lemma 9 and Theorem 5 (respectively)].

3. Standard elements in the dual of the lattice of T_1-topologies.

Following [**4**], an element a of a lattice L is called *standard* when

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y) \quad \text{for all } x, y \in L,$$

and a is called *distributive* when

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \quad \text{for all } x, y \in L.$$

It follows from Theorems 1 and 3 of **[4]** that any standard element is distributive and that all standard elements form a sublattice of L.

LEMMA 3.1. *Let a be an element of an atomistic lattice L. The following three statements are equivalent.*

(α) *a is standard.*

(β) *a is distributive.*

(γ) *If p is an atom of L such that $p \leqq a \vee x$ and $p \nleqq x$ then $p \leqq a$.*

PROOF. It is easy to verify the implications $(\beta) \Rightarrow (\gamma) \Rightarrow (\alpha)$, and the details are omitted.

Let X be an infinite set. We denote by $\mathscr{C}(X)$ the collection of all cofinite subsets of X. For any subset A of X, it is evident that $\mathscr{S}(A) = \mathscr{P}(A) \cup \mathscr{C}(X)$ is a T_1-topology. Especially, $\mathscr{S}(X)$ is the discrete topology, $\mathscr{S}(\varnothing)$ is the cofinite topology and $\mathscr{S}(\{x\})$ is an atom of $L_T(X)$ for any $x \in X$. The set $\{\mathscr{S}(A); A \subset X\}$ forms a Boolean sublattice of $L_T(X)$, which coincides with the lattice Λ_0 appeared in **[1]**.

THEOREM 3.1. *Let $\mathscr{T}_0$ be an element of the lattice $L_T(X)$ of T_1-topologies on an infinite set X, and let $\mathscr{T}_0 \neq \mathscr{S}(\varnothing)$. The following three statements are equivalent.*

(α) *$\mathscr{T}_0$ is standard in the dual of $L_T(X)$.*

(β) *$\mathscr{T}_0$ is distributive in the dual of $L_T(X)$.*

(γ) *$\mathscr{T}_0 = \mathscr{S}(X-F)$ for some finite subset F of X.*

PROOF. Since $L_T(X)$ is dual-atomistic, it follows from Lemma 3.1 that each of (α) and (β) is equivalent to the following statement:

(δ) If $\mathscr{T}(x, \mathscr{U}) \geqq \mathscr{T}_0 \wedge \mathscr{T}$ and $\mathscr{T}(x, \mathscr{U}) \ngeqq \mathscr{T}$ then $\mathscr{T}(x, \mathscr{U}) \geqq \mathscr{T}_0$.

First, we shall prove that (γ) implies (δ). Let $\mathscr{T}(x, \mathscr{U}) \geqq \mathscr{S}(X-F) \wedge \mathscr{T}$. If $\mathscr{T}(x, \mathscr{U}) \ngeqq \mathscr{T}$, then there exists $G \in \mathscr{T}$ with $G \notin \mathscr{T}(x, \mathscr{U})$, whence $x \in G \notin \mathscr{U}$. Since $G-F \in \mathscr{S}(X-F) \cap \mathscr{T} \subset \mathscr{T}(x, \mathscr{U})$ and $G-F \notin \mathscr{U}$, we have $G-F \subset X-\{x\}$, whence $x \in F$. Hence, $\mathscr{S}(X-F) \leqq \mathscr{T}(x, \mathscr{U})$. Therefore, $\mathscr{T}_0 = \mathscr{S}(X-F)$ satisfies (δ).

Next, we assume that $\mathscr{T}_0$ satisfies (δ), and we put $F = \{x \in X; \{x\} \notin \mathscr{T}_0\}$. We shall prove that $\mathscr{T}_0 = \mathscr{S}(X-F)$. Since $\{x\} \in \mathscr{T}_0$ for every $x \in X-F$, we have $\mathscr{P}(X-F) \subset \mathscr{T}_0$, whence $\mathscr{S}(X-F) \leqq \mathscr{T}_0$. If $\mathscr{S}(X-F) < \mathscr{T}_0$, then there would exist $G \in \mathscr{T}_0$ such that $G \notin \mathscr{S}(X-F)$. Then we have $G \cap F \neq \varnothing$. We take $x \in G \cap F$ and put $\mathscr{T} = \mathscr{S}(\{x\})$. Since $G \notin \mathscr{C}(X)$, the set $A = X-G$ is infinite. Hence, there exists a nonprincipal ultrafilter $\mathscr{U}$ on X such that $A \cup \{x\} \in \mathscr{U}$. Since $\{x\} \notin \mathscr{T}_0$, we have $\mathscr{T}_0 \wedge \mathscr{T} = \mathscr{S}(\varnothing) \leqq \mathscr{T}(x, \mathscr{U})$. Moreover, $\mathscr{T} \nleqq \mathscr{T}(x, \mathscr{U})$ since $\{x\} \in \mathscr{T}$. On the other hand, we have

$(A \cup \{x\}) \cap G = \{x\} \notin \mathscr{U}$ since $\mathscr{U}$ is nonprincipal. Since $A \cup \{x\} \in \mathscr{U}$, we have $G \notin \mathscr{U}$. Hence, $G \notin \mathscr{T}(x, \mathscr{U})$, and therefore $\mathscr{T}_0 \nleqq \mathscr{T}(x, \mathscr{U})$. This contradicts our assumption. Thus we get $\mathscr{T}_0 = \mathscr{S}(X-F)$.

We shall prove that F is finite. Since $\mathscr{T}_0 \neq \mathscr{S}(\varnothing)$, there exists $x \in X-F$. If F were infinite, then there would exist a subset A of F such that both A and $X-A$ are infinite. Put $\mathscr{T} = \mathscr{S}(\varnothing) \cup \{G \cap (A \cup \{x\}); G \in \mathscr{C}(X)\}$. It is evident that $\mathscr{T} \in L_T(X)$. Since $X-(A \cup \{x\})$ is infinite, there exists a nonprincipal ultrafilter $\mathscr{U}$ which contains this set. We have $\mathscr{T}_0 \nleqq \mathscr{T}(x, \mathscr{U})$ since $\{x\} \in \mathscr{T}_0$. Since $A \cup \{x\} \notin \mathscr{U}$, we have $A \cup \{x\} \notin \mathscr{T}(x, \mathscr{U})$. Hence, $\mathscr{T} \nleqq \mathscr{T}(x, \mathscr{U})$. If $G \in \mathscr{C}(X)$, then $G \cap (A \cup \{x\}) \cap F \supset G \cap A \neq \varnothing$, since A is infinite. Hence, $G \cap (A \cup \{x\}) \not\subset X-F$, whence $G \cap (A \cup \{x\}) \notin \mathscr{S}(X-F) = \mathscr{T}_0$. Therefore, $\mathscr{T}_0 \wedge \mathscr{T} = \mathscr{S}(\varnothing) \leqq \mathscr{T}(x, \mathscr{U})$. This contradicts our assumption. Thus, it has been proved that (δ) implies (γ). This completes the proof.

It is shown in [**2**, Chapter III, §9] that an element of a lattice L is neutral if and only if it is standard in both L and its dual and that if a neutral element has a complement then it is also neutral. Hence, it follows from the above theorem that

COROLLARY. *The lattice $L_T(X)$ has no neutral element except the greatest element $\mathscr{S}(X)$ and the least element $\mathscr{S}(\varnothing)$.*

Finally, we remark that the congruence relation $\mathscr{T}_1 \equiv \mathscr{T}_2$ (θ) in $L_T(X)$ defined in Theorem 2.2 coincides with the relation defined by each of the following equations:

$$\mathscr{T}_1 \wedge \mathscr{S}(X-F) = \mathscr{T}_2 \wedge \mathscr{S}(X-F), \quad \mathscr{T}_1 \vee \mathscr{S}(F) = \mathscr{T}_2 \vee \mathscr{S}(F).$$

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