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AN ENDOMORPHISM RING WHICH IS NOT ORE

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ABSTRACT. An example is given of a finitely generated module over a commutative ring whose endomorphism ring does not have a classical ring of quotients.

Vasconcelos [3] has shown that the endomorphism ring of a finitely generated module over a commutative noetherian ring has a classical ring of quotients. In [1] the author showed that the endomorphism ring of a finitely generated projective module over a commutative ring has a classical ring of quotients. The purpose of this note is to supply an example of a finitely generated module over a commutative ring whose endomorphism ring does not have a classical ring of quotients.

We recall that a ring A has a (right) classical ring of quotients Q if (i) $A\subseteq Q$, (ii) every regular element (nondivisor of zero) of A is invertible in Q, and (iii) every element of Q has the form ab^{-1} where a is an element of A and b is a regular element of A. A necessary and sufficient condition for a ring A to have a right ring of quotients is for A to satisfy the right Ore condition: Given a, b in A, b regular, then there exist a', b' in A, b' regular, such that $ab'=ba'$.

Let R be a commutative ring and I an ideal of R such that:

(i) whenever $a\in R$ is regular then $a+i$ is regular for all $i\in I$;

(ii) $\mathrm{Ann}_R(I)=\{x\in R\mid xI=0\}=0$;

(iii) there exists $z\in R$ with $\mathrm{Ann}_R(z)\neq 0$ but $\mathrm{Ann}_{R/I}(\bar z)=0$ where $\bar z=z+I\in R/I$. Let $M=R/I\oplus R$. M is clearly a finitely generated R-module. We show that $A=\mathrm{End}_R(M)$ does not have a right classical quotient ring. Observe that

$$A\approx\begin{bmatrix}\mathrm{End}_R(R/I) & \mathrm{Hom}_R(R, R/I)\\ \mathrm{Hom}_R(R/I, R) & \mathrm{End}_R(R)\end{bmatrix}.$$

Let $f\in\mathrm{Hom}_R(R/I, R)$ then $0=f(\bar i)=if(\bar 1)$ where $i\in I$; $\mathrm{Ann}_R(I)=0$ implies $f(\bar 1)=0$ and hence $f=0$. Thus $\mathrm{Hom}_R(R/I, R)=0$. Also

$$\mathrm{End}_R(R/I)\simeq\mathrm{End}_{R/I}(R/I)\simeq R/I\quad\text{and}\quad\mathrm{Hom}_R(R, R/I)\simeq R/I.$$

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Thus

$$A \approx \begin{bmatrix} R/I & R/I \\ 0 & R \end{bmatrix}.$$

If $\left[\begin{smallmatrix} \bar{a} & b \\ 0 & c \end{smallmatrix}\right]$ is regular in A then $\bar{a}$ is regular in R/I and c is regular in R. Simple computation shows $\left[\begin{smallmatrix} \bar{z} & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is regular in A, where z is the element of (iii) above.

We now show that A does not satisfy the right Ore condition, for if it did then given $\left[\begin{smallmatrix} \bar{z} & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & \bar{1} \\ 0 & 0 \end{smallmatrix}\right]$ there would exist $\left[\begin{smallmatrix} \bar{a} & b \\ 0 & c \end{smallmatrix}\right]$ regular in A and $\left[\begin{smallmatrix} \bar{u} & \bar{v} \\ 0 & w \end{smallmatrix}\right]$ in A such that

$$\begin{bmatrix} \bar{z} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{u} & \bar{v} \\ 0 & w \end{bmatrix} = \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & c \end{bmatrix},$$

which says, in particular, that $\bar{z}\bar{v}=\bar{c}$, hence $zv=c+i$ for some $i \in I$. By property (i) $c+i$ is regular since c is regular, thus zv is regular and so z is regular, a contradiction to the fact that $\mathrm{Ann}_R(z)\neq 0$.

It remains to show that there exists a ring R with properties (i), (ii), and (iii). For this purpose let F be a field and $S=F[X, Y]/(X^2, XY)$ where X and Y are (commuting) indeterminants. Let $N=(X)/(X^2, XY)$, $y=Y+(X^2, XY) \in S$, and $\bar{y}$ the image of y in S/N. We remark that $S/N \simeq F[Y]$ and $\bar{y}$ corresponds to Y under the isomorphism. $\mathrm{Ann}_S(y)\neq 0$ since $[Y+(X^2, XY)][X+(X^2, XY)]=(X^2, XY)$. However $\mathrm{Ann}_{S/N}(\bar{y})=0$, since $\mathrm{Ann}_{F[Y]}(Y)=0$. Since N is a nil ideal $a+n$ is regular for any regular element $a \in S$ and any $n \in N$. Since N is nilpotent we are still missing property (ii). We remedy this as follows: Let $x_1, x_2, x_3, \cdots$ be a countable set of (commuting) indeterminants and set

$$R = S[x_1, x_2, \cdots]/(x_1^2, x_2^3, x_3^4, \cdots),$$

where S is the ring constructed above. Let

$$I = (N \cdot S[x_1, x_2, \cdots] + (x_1, x_2, \cdots))/(x_1^2, x_2^3, \cdots).$$

Clearly every element of I is nilpotent and so property (i) holds. $\mathrm{Ann}_R(I)=0$ since if $a \in R$ such that $aI=0$ then a must have $x_1, x_2^2, x_3^3, \cdots$ all as factors, an impossibility. Finally let z be the image in R of $y \in S$. Since $\mathrm{Ann}_S(y)\neq 0$, say $sy=0$ for $0\neq s \in S$, then $\bar{s}=s+(x_1^2, x_2^3, \cdots)\neq 0$ in R and $\bar{s}z=0$, so $\mathrm{Ann}_R(z)\neq 0$. Let $\bar{z}=z+I \in R/I$; we show $\mathrm{Ann}_{R/I}(\bar{z})=0$. Suppose $\bar{g} \in R/I$ and $0 = \bar{z}\bar{g}$ where $g=s+h+(x_1^2, x_2^3, \cdots) \in R$ with $s \in S$ and $h \in (x_1, x_2, \cdots)$. Now $\bar{z}\bar{g}=0$ implies that $zg \in I$ that is $sy+yh \in N \cdot S[x_1, \cdots]+(x_1, \cdots)$, which says $sy \in N$ since $s, y \in S$. But $\mathrm{Ann}_{S/N}(\bar{y})=0$ so $s \in N$ and hence $g \in I$, so $\bar{g}=0$ and $\mathrm{Ann}_{R/I}(\bar{z})=0$.

As a final remark we observe that by Nagata's "principle of idealization" [2] we can form a new ring R^* such that M is an ideal of R^* and also $\mathrm{End}_R(M) \simeq \mathrm{End}_{R^*}(M)$. This fact and the last example show that one cannot hope to generalize "finitely generated projective" in [1] to "finitely generated torsionless."

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