

AN ENDOMORPHISM RING WHICH IS NOT ORE

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ABSTRACT. An example is given of a finitely generated module over a commutative ring whose endomorphism ring does not have a classical ring of quotients.

Vasconcelos [3] has shown that the endomorphism ring of a finitely generated module over a commutative noetherian ring has a classical ring of quotients. In [1] the author showed that the endomorphism ring of a finitely generated projective module over a commutative ring has a classical ring of quotients. The purpose of this note is to supply an example of a finitely generated module over a commutative ring whose endomorphism ring does not have a classical ring of quotients.

We recall that a ring A has a (right) classical ring of quotients Q if (i) $A \subseteq Q$, (ii) every regular element (nondivisor of zero) of A is invertible in Q , and (iii) every element of Q has the form ab^{-1} where a is an element of A and b is a regular element of A . A necessary and sufficient condition for a ring A to have a right ring of quotients is for A to satisfy the right Ore condition: Given a, b in A , b regular, then there exist a', b' in A , b' regular, such that $ab' = ba'$.

Let R be a commutative ring and I an ideal of R such that:

- (i) whenever $a \in R$ is regular then $a + i$ is regular for all $i \in I$;
- (ii) $\text{Ann}_R(I) = \{x \in R \mid xI = 0\} = 0$;
- (iii) there exists $z \in R$ with $\text{Ann}_R(z) \neq 0$ but $\text{Ann}_{R/I}(\bar{z}) = 0$ where $\bar{z} = z + I \in R/I$. Let $M = R/I \oplus R$. M is clearly a finitely generated R -module. We show that $A = \text{End}_R(M)$ does not have a right classical quotient ring. Observe that

$$A \approx \begin{bmatrix} \text{End}_R(R/I) & \text{Hom}_R(R, R/I) \\ \text{Hom}_R(R/I, R) & \text{End}_R(R) \end{bmatrix}.$$

Let $f \in \text{Hom}_R(R/I, R)$ then $0 = f(i) = f(\bar{1})$ where $i \in I$; $\text{Ann}_R(I) = 0$ implies $f(\bar{1}) = 0$ and hence $f = 0$. Thus $\text{Hom}_R(R/I, R) = 0$. Also

$$\text{End}_R(R/I) \simeq \text{End}_{R/I}(R/I) \simeq R/I \quad \text{and} \quad \text{Hom}_R(R, R/I) \simeq R/I.$$

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Thus

$$A \approx \begin{bmatrix} R/I & R/I \\ 0 & R \end{bmatrix}.$$

If $\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix}$ is regular in A then \bar{a} is regular in R/I and \bar{c} is regular in R . Simple computation shows $\begin{bmatrix} \bar{z} & 0 \\ 0 & \bar{1} \end{bmatrix}$ is regular in A , where z is the element of (iii) above.

We now show that A does not satisfy the right Ore condition, for if it did then given $\begin{bmatrix} \bar{z} & 0 \\ 0 & \bar{1} \end{bmatrix}$ and $\begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix}$ there would exist $\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix}$ regular in A and $\begin{bmatrix} \bar{u} & \bar{v} \\ 0 & \bar{w} \end{bmatrix}$ in A such that

$$\begin{bmatrix} \bar{z} & 0 \\ 0 & \bar{1} \end{bmatrix} \begin{bmatrix} \bar{u} & \bar{v} \\ 0 & \bar{w} \end{bmatrix} = \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix},$$

which says, in particular, that $\bar{z}\bar{v} = \bar{c}$, hence $zv = c + i$ for some $i \in I$. By property (i) $c + i$ is regular since c is regular, thus zv is regular and so z is regular, a contradiction to the fact that $\text{Ann}_R(z) \neq 0$.

It remains to show that there exists a ring R with properties (i), (ii), and (iii). For this purpose let F be a field and $S = F[X, Y]/(X^2, XY)$ where X and Y are (commuting) indeterminants. Let $N = (X)/(X^2, XY)$, $y = Y + (X^2, XY) \in S$, and \bar{y} the image of y in S/N . We remark that $S/N \cong F[Y]$ and \bar{y} corresponds to Y under the isomorphism. $\text{Ann}_S(y) \neq 0$ since $[Y + (X^2, XY)][X + (X^2, XY)] = (X^2, XY)$. However $\text{Ann}_{S/N}(\bar{y}) = 0$, since $\text{Ann}_{F[Y]}(Y) = 0$. Since N is a nil ideal $a + n$ is regular for any regular element $a \in S$ and any $n \in N$. Since N is nilpotent we are still missing property (ii). We remedy this as follows: Let x_1, x_2, x_3, \dots be a countable set of (commuting) indeterminants and set

$$R = S[x_1, x_2, \dots]/(x_1^2, x_2^3, x_3^4, \dots),$$

where S is the ring constructed above. Let

$$I = (N \cdot S[x_1, x_2, \dots] + (x_1, x_2, \dots))/(x_1^2, x_2^3, \dots).$$

Clearly every element of I is nilpotent and so property (i) holds. $\text{Ann}_R(I) = 0$ since if $a \in R$ such that $aI = 0$ then a must have x_1, x_2^3, x_3^4, \dots all as factors, an impossibility. Finally let z be the image in R of $y \in S$. Since $\text{Ann}_S(y) \neq 0$, say $sy = 0$ for $0 \neq s \in S$, then $\bar{s} = s + (x_1^2, x_2^3, \dots) \neq 0$ in R and $\bar{s}z = 0$, so $\text{Ann}_R(z) \neq 0$. Let $\bar{z} = z + I \in R/I$; we show $\text{Ann}_{R/I}(\bar{z}) = 0$. Suppose $\bar{g} \in R/I$ and $0 = \bar{z}\bar{g}$ where $g = s + h + (x_1^2, x_2^3, \dots) \in R$ with $s \in S$ and $h \in (x_1, x_2, \dots)$. Now $\bar{z}\bar{g} = 0$ implies that $zg \in I$ that is $sy + yh \in N \cdot S[x_1, \dots] + (x_1, \dots)$, which says $sy \in N$ since $s, y \in S$. But $\text{Ann}_{S/N}(\bar{y}) = 0$ so $s \in N$ and hence $g \in I$, so $\bar{g} = 0$ and $\text{Ann}_{R/I}(\bar{z}) = 0$.

As a final remark we observe that by Nagata's "principle of idealization" [2] we can form a new ring R^* such that M is an ideal of R^* and also $\text{End}_R(M) \simeq \text{End}_{R^*}(M)$. This fact and the last example show that one cannot hope to generalize "finitely generated projective" in [1] to "finitely generated torsionless."

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