PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

THE RADIUS OF STARLIKENESS OF CERTAIN ANALYTIC FUNCTIONS

P. L. BAJPAI AND PREM SINGH

ABSTRACT. Let $f(z)=\frac{1}{2}[zF(z)]'$, where $F(z)$ is starlike of order α, $0\leqq\alpha<1$ in $D=\{z:|z|<1\}$. The purpose of this paper is to find out the disc in which $f(z)$ is starlike of order β, $0\leqq\beta<1$. Results are best possible.

1. **Introduction.** Let S denote the class of all regular and univalent functions $f(z)$ in $D=\{z:|z|<1\}$ which are normalized by the conditions $f(0)=0$ and $f'(0)=1$. For a fixed α, $0\leqq\alpha<1$, let $S^*(\alpha)$ denote the subclass of S, consisting of all functions f satisfying the condition

$$\operatorname{Re}\{zf'(z)/f(z)\} > \alpha \quad \text{for } z\in D. \tag{1.1}$$

Functions in the class $S^*(\alpha)$ are known as starlike functions of order α.

In a recent paper R. J. Libera and A. E. Livingston [1] determined the disc in which

$$f(z) = \tfrac{1}{2}[zF(z)]' \tag{1.2}$$

(where $F(z)$ is a starlike function of order α, $0\leqq\alpha<1$, for $z\in D$) is starlike of order β if, $0\leqq\alpha\leqq\beta<1$. They were unable to obtain suitable results for the complementary case, that is when $\beta<\alpha$. The purpose of the paper is to give a method which covers both the cases.

Recently V. Singh and R. M. Goel [5] determined the disc in which $f(z)$ given by (1.2) is starlike. For proving the above result, they proved a lemma whose proof is independent of the variational techniques of Robertson [3] and Sakaguchi [4]. Their technique is more powerful than the variational techniques.

In this paper we prove the same lemma by using a similar technique and claim that our technique is more powerful than the technique given in [5]. It is because this technique succeeds in proving similar results

Received by the editors December 15, 1972 and, in revised form, August 27, 1973.
AMS (MOS) subject classifications (1970). Primary 30A32.
Key words and phrases. Univalent functions, starlike functions of order α, radius of starlikeness.

© American Mathematical Society 1974

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

for functions of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{kn+1} z^{kn+1}, \qquad n \geqq 1,$$

while their technique fails.

2. Statement and proof of the Lemma.

LEMMA. *Let α satisfy $0 \leqq \alpha < 1$. Let $r(\alpha)$ denote the root, unique in $(\frac{1}{2}, 1]$ of the equation*

$$\alpha(2\alpha - 1)r^3 + \alpha(7 - 2\alpha)r^2 + (5 - 4\alpha)r - 3 = 0. \tag{2.1}$$

If $f(z)$ is given by (1.2), *then for $|z| = r$, $0 \leqq r < 1$,*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{2}{1-\alpha}\{((1+\alpha)(4-2\alpha)a)^{1/2} - 1 - a\} \quad \textit{for } r(\alpha) \leqq r < 1,$$

$$\geqq \frac{1 - (2 - 4\alpha)r + \alpha(2\alpha - 1)r^2}{(1+r)(1+\alpha r)} \quad \textit{for } 0 \leqq r \leqq r(\alpha), \tag{2.2}$$

where $a = (1 - \alpha r^2)/(1 - r^2)$. This result is sharp.

PROOF. Since $F(z) \in S^*(\alpha)$, there exists a function $w(z) = z\phi(z)$ satisfying Schwarz's lemma such that

$$\frac{zF'(z)}{F(z)} = \frac{1 + (2\alpha - 1)z\phi(z)}{1 + z\phi(z)} \quad \text{for } z \in D. \tag{2.3}$$

From (1.2) and (2.3) we have

$$f(z) = F(z)\left(\frac{1 + \alpha z\phi(z)}{1 + z\phi(z)}\right). \tag{2.4}$$

Differentiating (2.4) logarithmically with respect to z and combining it with (2.3), we get

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)\left\{\frac{1 - z\phi(z)}{1 + z\phi(z)} - \frac{z^2\phi'(z) + z\phi(z)}{(1 + \alpha z\phi(z))(1 + z\phi(z))}\right\}$$

$$= (2\alpha - 1) + \frac{3 - 2\alpha}{1 + z\phi(z)} - \frac{1}{1 + \alpha z\phi(z)} - \frac{(1-\alpha)z^2\phi'(z)}{(1 + \alpha z\phi(z))(1 + z\phi(z))}.$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 2\alpha - 1 + (3 - 2\alpha)\operatorname{Re}\left\{\frac{1}{1 + z\phi(z)}\right\} - \operatorname{Re}\left\{\frac{1}{1 + \alpha z\phi(z)}\right\}$$

$$- \frac{(1-\alpha)(|z|^2 - |z\phi(z)|^2)}{(1 - |z|^2)\,|1 + z\phi(z)|\,|1 + \alpha z\phi(z)|}. \tag{2.5}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

The last inequality has been obtained by using the well-known result [7, p. 168] $|\phi'(z)|\leqq(1-|\phi(z)|^2)/(1-|z|^2)$.

Let $|z|=r$, $1/(1+z\phi(z))=u+iv$; then u satisfies

$$|u - 1/(1 - |z|^2)| \leqq |z|/(1 - |z|^2).$$

Then from (2.5) we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &\geqq 2\alpha - 1 + (3 - 2\alpha)u \\ &\quad - \frac{\alpha u + (1-\alpha)u^2 + (1-\alpha)v^2}{(\alpha + (1-\alpha)u)^2 + (1-\alpha)^2 v} \\ &\quad - \frac{(1-\alpha)(r^2(u^2+v^2) - (1-u)^2 - v^2)}{(1-r^2)((\alpha+(1-\alpha)u)^2 + (1-\alpha)^2v^2)^{1/2}} \\ &\equiv G(u, v, r) \quad \text{(say).}\end{aligned} \tag{2.6}$$

Differentiating G partially with respect to v, we have

$$\begin{aligned}\frac{\partial G}{\partial v} = v\Bigg[&\frac{-2\alpha(1-\alpha)(\alpha+(1-\alpha)u)}{\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^2} \\ &+ \frac{(1-\alpha)\{2(1-r^2)((\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2) + (r^2(u^2+v^2)-(1-u)^2-v^2)(1-\alpha)^2\}}{(1-r^2)((\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2)^{3/2}}\Bigg].\end{aligned}$$

The minimum of G will occur at $v=0$ if

$$\begin{aligned}&\Bigg[\frac{-2\alpha(1-\alpha)(\alpha+(1-\alpha)u)}{\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^2} + \frac{2(1-\alpha)}{((\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2)^{1/2}} \\ &\qquad + \frac{\{r^2(v^2+u^2)-(1-u)^2-v^2\}(1-\alpha)^3}{(1-r^2)\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^{3/2}}\Bigg] \\ &> 0.\end{aligned} \tag{2.7}$$

Since $|z\phi(z)|\leqq r$ for $|z|=r$, and $|z\phi(2)|=((1-u)^2+v^2)/(u^2+v^2)$ we can easily conclude that

$$r^2(v^2 + u^2) - (1 - u)^2 - v^2 \geqq 0, \tag{2.8}$$

and from $|u-1/(1-|z|^2)|\leqq|z|/(1-|z|^2)$, we have

$$1/(1 + r) \leqq u \leqq 1/(1 - r). \tag{2.9}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Therefore using (2.8) in (2.7) we have

$$\left[\frac{-2\alpha(1-\alpha)(\alpha+(1-\alpha)u)}{\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^2}+\frac{2(1-\alpha)}{((\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2)^{1/2}}\right.$$
$$\left.+\frac{(r^2(v^2+u^2)-(1-u)^2-v^2)(1-\alpha)^3}{(1-r^2)\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^{3/2}}\right]$$
$$\geqq 2(1-\alpha)\frac{\{-\alpha+\alpha^2+2\alpha(1-\alpha)u+(1-\alpha)^2u^2\}(\alpha+(1-\alpha)u)}{\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^2}.$$

But in view of (2.9),

$$-\alpha+\alpha^2+2\alpha(1-\alpha)u+(1-\alpha)^2u^2\geqq(1-\alpha)(1-\alpha r^2)/(1+r)^2>0.$$

Therefore the minimum of G occurs at $v=0$. On putting $v=0$ in (2.6), we obtain

$$\begin{aligned}g(u,r)&\equiv G(u,0,r)\\&=(2\alpha-1)+(3-2\alpha)u-\frac{u}{\alpha+(1-\alpha)u}\\&\quad-\frac{(r_2u_2-(1-u)^2)(1-\alpha)}{(1-r^2)(\alpha+(1-\alpha)u)}\\&=2\alpha-1-\frac{1+\alpha}{1-\alpha}-\frac{2}{(1-r^2)}+(4-2\alpha)u\\&\quad+\frac{(1+\alpha)(1-\alpha r^2)}{(1-r^2)(1-\alpha)(\alpha+(1-\alpha)u)}.\end{aligned}$$

The absolute minimum of g in $(0,\infty)$ is attained at

$$\alpha+(1-\alpha)u_0=\left(\frac{(1+\alpha)(1-\alpha r^2)}{2(2-\alpha)(1-r^2)}\right)^{1/2}=\left(\frac{(1+\alpha)a}{4-2\alpha}\right)^{1/2},$$

i.e.

$$u_0=\frac{1}{1-\alpha}\left\{\left(\frac{(1+\alpha)a}{4-2\alpha}\right)^{1/2}-\alpha\right\}, \tag{2.10}$$

where $a=(1-\alpha r^2)/(1-r^2)$, provided $u_0\in[1/(1+r),1/(1-r)]$ and is equal to

$$g(u_0,r)\equiv(2/(1-\alpha))\{((1+\alpha)(4-2\alpha)a)^{1/2}-1-a\}. \tag{2.11}$$

It is easy to check that $u_0<1/(1-r)$, but it is not always greater than $1/(1+r)$. In such a case when $u_0\notin[1/(1+r),1/(1-r)]$, one can easily

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

check that the minimum of $g(u, r)$ on the segment $[1/(1+r), 1/(1-r)]$ is attained at

$$u_1 = 1/(1 + r) \tag{2.12}$$

and is equal to

$$g(u_1, r) \equiv \frac{1 - (2 - 4\alpha)r + \alpha(2\alpha - 1)r^2}{(1 + r)(1 + \alpha r)}; \qquad g(u_1, r) = g(u_0,r) \tag{2.13}$$

for such values of α for which $u_0=u_1$, i.e.

$$\frac{1}{(1 - \alpha)}\left\{\left(\frac{(1 + \alpha)(1 - \alpha r^2)}{(4 - 2\alpha)(1 - r^2)}\right)^{1/2} - \alpha\right\} = \frac{1}{1 + r}. \tag{2.14}$$

On simplifying (2.14) we get

$$g_1(\alpha, r) \equiv \alpha(2\alpha - 1)r^3 + \alpha(7 - 2\alpha)r^2 + (5 - 4\alpha)r - 3 = 0.$$

It is easy to check that $g_1(\alpha, r)$ is a strictly increasing function of r for $0 \leqq r \leqq 1$, for each α

$$g_1(\alpha, \tfrac{1}{2}) = -\frac{\alpha^2}{4} - \frac{3}{8}\alpha - \frac{1}{2} < 0 \quad \text{and} \quad g_1(\alpha, 1) = 2(1 - \alpha) > 0.$$

Thus $g_1(\alpha, r)$ has an unique root $r(\alpha)$ in $(\frac{1}{2}, 1]$. The equality sign in the first inequality of (2.2) is attained for the function

$$F(z) = z(1 - 2bz + z^2)^{-1+\alpha}, \tag{2.15}$$

where the constant b is determined from

$$\frac{1 - (1 + \alpha)br + \alpha r^2}{1 - 2br + r^2} = \left(\frac{(1 + \alpha)a}{4 - 2\alpha}\right)^{1/2} \equiv R_0 \quad \text{(say)}. \tag{2.16}$$

It can be easily checked that $F(z)$ given by (2.15) is regular in D because 'b' lies in the interval $(-1, 1)$. This can be done by showing that

$$\mu_b(r) = \frac{1 - (1 + \alpha)br + \alpha r^2}{1 - 2br + r^2} > \left(\frac{(1 + \alpha)a}{4 - 2\alpha}\right)^{1/2} = R_0 \tag{2.17}$$

when $b=-1$ for $0 \leqq r < 1$, and that

$$\mu_b(r) < R_0 \tag{2.18}$$

when $b=-1$ for $r > r(\alpha)$, where $r(\alpha)$ is given by (2.1).

(2.17) at $b=1$ becomes

$$(1 - \alpha r)^2/(1 - r) > (1 + \alpha)(1 - \alpha r^2)/(4 - 2\alpha)(1 + r). \tag{2.19}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

But $(1-\alpha r)/(1-r) \geqq 1$. Therefore (2.19) is satisfied if

$$(4-2\alpha)(1-\alpha)r+3(1-\alpha)(1-\alpha r^2)>0. \tag{2.20}$$

(2.20) is true for $0 \leqq r < 1$. (2.18) at $b=-1$ becomes

$$(1+\alpha r)^2/(1+r) < (1+\alpha)(1-\alpha r^2)/(4-2\alpha)(1-r). \tag{2.21}$$

(2.21) is true for $r > r(\alpha)$, where $r(\alpha)$ is given by (2.1). From (2.15) we have

$$\frac{zF'(z)}{F(z)} = \frac{1-2\alpha bz+(2\alpha-1)z^2}{1-2bz+z^2}. \tag{2.22}$$

Therefore from (2.22) and (1.2) we get

$$f(z) = F(z)\left(\frac{1-(1+\alpha)bz+\alpha z^2}{1-2bz+z^2}\right).$$

Differentiating $f(z)$ logarithmically with respect to z, we obtain

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{zF'(z)}{F(z)} + \frac{z(1-\alpha)(b-2z+bz^2)}{(1-(1+\alpha)bz+\alpha z^2)(1-2bz+z^2)}\\ &= 2R_0-1+(a-R_0)(1+\alpha-2R_0)/R_0(1-\alpha)\\ &= (2/(1-\alpha))\{((1+\alpha)(4-2\alpha)a)^{1/2}-1-a\} \quad \text{at } z=r.\end{aligned}$$

The equality sign in the second inequality of (2.2) is attained for the function

$$F(z) = z(1+z)^{-2+2\alpha} \quad \text{at } z=r. \tag{2.23}$$

From (2.23) and (1.2) we get $f(z)=F(z)(1+\alpha z)/(1+z)$,

$$\therefore \frac{zf'(z)}{f(z)} = \frac{1-(2-4\alpha)z+\alpha(2\alpha-1)z^2}{(1+z)(1+\alpha z)}.$$

Hence the proof of the lemma is complete.

3. Statement and proof of the main result.

THEOREM. *Let $f(z)$ be given by* (1.2) *and let $r(\alpha)$ be the root, unique in* $(\frac{1}{2}, 1]$, *of the equation* (2.1). *Then $f(z)$ is starlike of order β for $|z|<r_0$, where r_0 is the smallest positive root of the equation*

$$1-\beta+(2(2\alpha-1)-\beta(1+\alpha))r+\alpha(2\alpha-\beta-1)r^2=0 \quad \text{if } 0 \leqq r_0 \leqq r(\alpha), \tag{3.1}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

and of the equation

$$a^2 + \{\beta(1-\alpha) - 2(1+\alpha-\alpha^2)\}a + \{1+\beta(1-\alpha)/2\}^2 = 0 \tag{3.2}$$
$$\textit{if } r(\alpha) \leqq r_0 < 1,$$

where $a=(1-\alpha r^2)/(1-r^2)$. *This result is sharp.*

PROOF. Since $f(z)$ is as stated in the Lemma,

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \\
&\qquad \geqq \frac{1-(2-4\alpha)r+\alpha(2\alpha-1)r^2}{(1+r)(1+\alpha r)} - \beta \\
&\qquad = \frac{1-\beta+(2(2\alpha-1)-\beta(1+\alpha))r+\alpha(2\alpha-\beta-1)r^2}{(1+r)(1+\alpha r)} \\
&\qquad\qquad \text{if } 0 \leqq r \leqq r(\alpha),
\end{aligned} \tag{3.3}$$

and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geqq \frac{2}{1-\alpha}\left\{((1+\alpha)(4-2\alpha)a)^{1/2} - 1 - a - \frac{\beta(1-\alpha)}{2}\right. \tag{3.4}$$
$$\text{if } r(\alpha) \leqq r < 1.$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geqq 0 \tag{3.5}$$
$$\text{if } (1-\beta)+(2(2\alpha-1)-\beta(1+\alpha))r+\alpha(2\alpha-\beta-1)r^2 \geqq 0,$$

and

$$a^2 + \{\beta(1-\alpha) - 2(1+\alpha-\alpha^2)\}a + \{1+\beta(1-\alpha)/2\}^2 \geqq 0. \tag{3.6}$$

(3.5) is valid only when $0\leqq r\leqq r_0\leqq r(\alpha)$, and (3.6) is valid only when $r(\alpha)\leqq r\leqq r_0<1$.

The equality in (3.3) is attained for the function given by (2.23), and that in (3.4) for the function given by (2.15).

By taking $\beta=0$ in (3.1) and (3.2), we obtain Theorem 3.2 of V. Singh and R. M. Goel [**5**] as a corollary of the above theorem.

One can easily see that Theorem 1 of Libera and Livingston [**1**] follows from (3.1) which they obtain under the restriction $\alpha\leqq\beta<1$. Thus, if $\beta=0$, α has to be zero. In our case if $\beta=0$, α need not be zero.

REFERENCES

1. R. J. Libera and A. E. Livingston, *On the univalence of some classes of regular functions*, Proc. Amer. Math. Soc. **30** (1971), 327–336. MR **44** #5442.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

2. A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **17** (1966), 352–357. MR **32** #5861.

3. M. S. Robertson, *Variational method for functions with positive real part*, Trans. Amer. Math. Soc. **102** (1962), 82–93. MR **24** #A3288.

4. K. Sakaguchi, *A variational method for functions with positive real part*, J. Math. Soc. Japan **16** (1964), 287–297. MR **31** #1375.

5. V. Singh and R. M. Goel, *On radii of convexity and starlikeness of some classes of functions*, J. Math. Soc. Japan **23** (1971), 323–339. MR **43** #7617.

6. V. A. Zmorovič, *On bounds of convexity for starlike functions of order* α *in the circle* $|z|<1$ *and in the circular region* $0<|z|<1$, Mat. Sb. (N.S.) **(68) 110** (1965), 65–70. (Russian) MR **33** #5875.

7. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.

Department of Mathematics, Indian Institute of Technology, Kanpur-208016, India

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use