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THE RADIUS OF STARLIKENESS OF CERTAIN ANALYTIC FUNCTIONS

P. L. BAJPAI AND PREM SINGH

ABSTRACT. Let $f(z)=\frac{1}{2}[zF(z)]'$, where $F(z)$ is starlike of order α, $0\leqq\alpha<1$ in $D=\{z:|z|<1\}$. The purpose of this paper is to find out the disc in which $f(z)$ is starlike of order β, $0\leqq\beta<1$. Results are best possible.

1. **Introduction.** Let S denote the class of all regular and univalent functions $f(z)$ in $D=\{z:|z|<1\}$ which are normalized by the conditions $f(0)=0$ and $f'(0)=1$. For a fixed α, $0\leqq\alpha<1$, let $S^*(\alpha)$ denote the subclass of S, consisting of all functions f satisfying the condition

$$\operatorname{Re}\{zf'(z)/f(z)\} > \alpha \quad \text{for } z \in D. \tag{1.1}$$

Functions in the class $S^*(\alpha)$ are known as starlike functions of order α.

In a recent paper R. J. Libera and A. E. Livingston [1] determined the disc in which

$$f(z) = \tfrac{1}{2}[zF(z)]' \tag{1.2}$$

(where $F(z)$ is a starlike function of order α, $0\leqq\alpha<1$, for $z\in D$) is starlike of order β if, $0\leqq\alpha\leqq\beta<1$. They were unable to obtain suitable results for the complementary case, that is when $\beta<\alpha$. The purpose of the paper is to give a method which covers both the cases.

Recently V. Singh and R. M. Goel [5] determined the disc in which $f(z)$ given by (1.2) is starlike. For proving the above result, they proved a lemma whose proof is independent of the variational techniques of Robertson [3] and Sakaguchi [4]. Their technique is more powerful than the variational techniques.

In this paper we prove the same lemma by using a similar technique and claim that our technique is more powerful than the technique given in [5]. It is because this technique succeeds in proving similar results

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for functions of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{kn+1} z^{kn+1}, \qquad n \geqq 1,$$

while their technique fails.

2. Statement and proof of the Lemma.

LEMMA. *Let α satisfy $0 \leqq \alpha < 1$. Let $r(\alpha)$ denote the root, unique in $(\frac{1}{2}, 1]$ of the equation*

$$\alpha(2\alpha - 1)r^3 + \alpha(7 - 2\alpha)r^2 + (5 - 4\alpha)r - 3 = 0. \tag{2.1}$$

If $f(z)$ is given by (1.2), *then for $|z|=r$, $0 \leqq r < 1$,*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{2}{1-\alpha}\{((1+\alpha)(4-2\alpha)a)^{1/2} - 1 - a\} \quad \text{for } r(\alpha) \leqq r < 1,$$

$$\geqq \frac{1 - (2 - 4\alpha)r + \alpha(2\alpha - 1)r^2}{(1+r)(1+\alpha r)} \quad \text{for } 0 \leqq r \leqq r(\alpha), \tag{2.2}$$

where $a = (1 - \alpha r^2)/(1 - r^2)$. This result is sharp.

PROOF. Since $F(z) \in S^*(\alpha)$, there exists a function $w(z) = z\phi(z)$ satisfying Schwarz's lemma such that

$$\frac{zF'(z)}{F(z)} = \frac{1 + (2\alpha - 1)z\phi(z)}{1 + z\phi(z)} \quad \text{for } z \in D. \tag{2.3}$$

From (1.2) and (2.3) we have

$$f(z) = F(z)\left(\frac{1 + \alpha z\phi(z)}{1 + z\phi(z)}\right). \tag{2.4}$$

Differentiating (2.4) logarithmically with respect to z and combining it with (2.3), we get

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)\left\{\frac{1 - z\phi(z)}{1 + z\phi(z)} - \frac{z^2\phi'(z) + z\phi(z)}{(1 + \alpha z\phi(z))(1 + z\phi(z))}\right\}$$

$$= (2\alpha - 1) + \frac{3 - 2\alpha}{1 + z\phi(z)} - \frac{1}{1 + \alpha z\phi(z)} - \frac{(1-\alpha)z^2\phi'(z)}{(1 + \alpha z\phi(z))(1 + z\phi(z))}.$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 2\alpha - 1 + (3 - 2\alpha)\operatorname{Re}\left\{\frac{1}{1 + z\phi(z)}\right\} - \operatorname{Re}\left\{\frac{1}{1 + \alpha z\phi(z)}\right\}$$

$$- \frac{(1-\alpha)(|z|^2 - |z\phi(z)|^2)}{(1 - |z|^2)\,|1 + z\phi(z)|\,|1 + \alpha z\phi(z)|}. \tag{2.5}$$

The last inequality has been obtained by using the well-known result [7, p. 168] $|\phi'(z)|\leqq(1-|\phi(z)|^2)/(1-|z|^2)$.

Let $|z|=r$, $1/(1+z\phi(z))=u+iv$; then u satisfies

$$|u - 1/(1 - |z|^2)| \leqq |z|/(1 - |z|^2).$$

Then from (2.5) we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &\geqq 2\alpha - 1 + (3 - 2\alpha)u \\ &\quad - \frac{\alpha u + (1-\alpha)u^2 + (1-\alpha)v^2}{(\alpha + (1-\alpha)u)^2 + (1-\alpha)^2 v} \\ &\quad - \frac{(1-\alpha)(r^2(u^2+v^2) - (1-u)^2 - v^2)}{(1-r^2)((\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2)^{1/2}} \\ &\equiv G(u, v, r) \quad \text{(say)}.\end{aligned} \tag{2.6}$$

Differentiating G partially with respect to v, we have

$$\begin{aligned}\frac{\partial G}{\partial v} = v\Bigg[&\frac{-2\alpha(1-\alpha)(\alpha + (1-\alpha)u)}{\{(\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2\}^2} \\ &+ (1-\alpha)\{2(1-r^2)((\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2) \\ &\frac{\qquad + (r^2(u^2+v^2) - (1-u)^2 - v^2)(1-\alpha)^2\}}{(1-r^2)((\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2)^{3/2}}\Bigg].\end{aligned}$$

The minimum of G will occur at $v=0$ if

$$\begin{aligned}&\Bigg[\frac{-2\alpha(1-\alpha)(\alpha + (1-\alpha)u)}{\{(\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2\}^2} + \frac{2(1-\alpha)}{((\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2)^{1/2}} \\ &\qquad + \frac{\{r^2(v^2+u^2) - (1-u)^2 - v^2\}(1-\alpha)^3}{(1-r^2)\{(\alpha + (1-\alpha)u)^2 + (1-\alpha)^2v^2\}^{3/2}}\Bigg] \\ &> 0.\end{aligned} \tag{2.7}$$

Since $|z\phi(z)|\leqq r$ for $|z|=r$, and $|z\phi(2)|=((1-u)^2+v^2)/(u^2+v^2)$ we can easily conclude that

$$r^2(v^2 + u^2) - (1-u)^2 - v^2 \geqq 0, \tag{2.8}$$

and from $|u-1/(1-|z|^2)|\leqq|z|/(1-|z|^2)$, we have

$$1/(1 + r) \leqq u \leqq 1/(1 - r). \tag{2.9}$$

Therefore using (2.8) in (2.7) we have

$$\left[\frac{-2\alpha(1-\alpha)(\alpha+(1-\alpha)u)}{\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^2}+\frac{2(1-\alpha)}{((\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2)^{1/2}}\right.$$
$$\left.+\frac{(r^2(v^2+u^2)-(1-u)^2-v^2)(1-\alpha)^3}{(1-r^2)\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^{3/2}}\right]$$
$$\geqq 2(1-\alpha)\frac{\{-\alpha+\alpha^2+2\alpha(1-\alpha)u+(1-\alpha)^2u^2\}(\alpha+(1-\alpha)u)}{\{(\alpha+(1-\alpha)u)^2+(1-\alpha)^2v^2\}^2}.$$

But in view of (2.9),

$$-\alpha+\alpha^2+2\alpha(1-\alpha)u+(1-\alpha)^2u^2\geqq(1-\alpha)(1-\alpha r^2)/(1+r)^2>0.$$

Therefore the minimum of G occurs at $v=0$. On putting $v=0$ in (2.6), we obtain

$$\begin{aligned} g(u,r) &\equiv G(u,0,r)\\ &=(2\alpha-1)+(3-2\alpha)u-\frac{u}{\alpha+(1-\alpha)u}\\ &\quad-\frac{(r_2u_2-(1-u)^2)(1-\alpha)}{(1-r^2)(\alpha+(1-\alpha)u)}\\ &=2\alpha-1-\frac{1+\alpha}{1-\alpha}-\frac{2}{(1-r^2)}+(4-2\alpha)u\\ &\quad+\frac{(1+\alpha)(1-\alpha r^2)}{(1-r^2)(1-\alpha)(\alpha+(1-\alpha)u)}. \end{aligned}$$

The absolute minimum of g in $(0,\infty)$ is attained at

$$\alpha+(1-\alpha)u_0=\left(\frac{(1+\alpha)(1-\alpha r^2)}{2(2-\alpha)(1-r^2)}\right)^{1/2}=\left(\frac{(1+\alpha)a}{4-2\alpha}\right)^{1/2},$$

i.e.

$$u_0=\frac{1}{1-\alpha}\left\{\left(\frac{(1+\alpha)a}{4-2\alpha}\right)^{1/2}-\alpha\right\}, \tag{2.10}$$

where $a=(1-\alpha r^2)/(1-r^2)$, provided $u_0\in[1/(1+r),1/(1-r)]$ and is equal to

$$g(u_0,r)\equiv(2/(1-\alpha))\{((1+\alpha)(4-2\alpha)a)^{1/2}-1-a\}. \tag{2.11}$$

It is easy to check that $u_0<1/(1-r)$, but it is not always greater than $1/(1+r)$. In such a case when $u_0\notin[1/(1+r),1/(1-r)]$, one can easily check that the minimum of $g(u, r)$ on the segment $[1/(1+r), 1/(1-r)]$ is attained at

$$u_1 = 1/(1+r) \tag{2.12}$$

and is equal to

$$g(u_1, r) \equiv \frac{1-(2-4\alpha)r+\alpha(2\alpha-1)r^2}{(1+r)(1+\alpha r)}; \qquad g(u_1, r) = g(u_0, r) \tag{2.13}$$

for such values of α for which $u_0 = u_1$, i.e.

$$\frac{1}{(1-\alpha)}\left\{\left(\frac{(1+\alpha)(1-\alpha r^2)}{(4-2\alpha)(1-r^2)}\right)^{1/2} - \alpha\right\} = \frac{1}{1+r}. \tag{2.14}$$

On simplifying (2.14) we get

$$g_1(\alpha, r) \equiv \alpha(2\alpha-1)r^3 + \alpha(7-2\alpha)r^2 + (5-4\alpha)r - 3 = 0.$$

It is easy to check that $g_1(\alpha, r)$ is a strictly increasing function of r for $0 \leqq r \leqq 1$, for each α

$$g_1(\alpha, \tfrac{1}{2}) = -\frac{\alpha^2}{4} - \frac{3}{8}\alpha - \frac{1}{2} < 0 \quad \text{and} \quad g_1(\alpha, 1) = 2(1-\alpha) > 0.$$

Thus $g_1(\alpha, r)$ has an unique root $r(\alpha)$ in $(\frac{1}{2}, 1]$. The equality sign in the first inequality of (2.2) is attained for the function

$$F(z) = z(1-2bz+z^2)^{-1+\alpha}, \tag{2.15}$$

where the constant b is determined from

$$\frac{1-(1+\alpha)br+\alpha r^2}{1-2br+r^2} = \left(\frac{(1+\alpha)a}{4-2\alpha}\right)^{1/2} \equiv R_0 \quad \text{(say)}. \tag{2.16}$$

It can be easily checked that $F(z)$ given by (2.15) is regular in D because 'b' lies in the interval $(-1, 1)$. This can be done by showing that

$$\mu_b(r) = \frac{1-(1+\alpha)br+\alpha r^2}{1-2br+r^2} > \left(\frac{(1+\alpha)a}{4-2\alpha}\right)^{1/2} = R_0 \tag{2.17}$$

when $b=-1$ for $0 \leqq r < 1$, and that

$$\mu_b(r) < R_0 \tag{2.18}$$

when $b=-1$ for $r > r(\alpha)$, where $r(\alpha)$ is given by (2.1).

(2.17) at $b=1$ becomes

$$(1-\alpha r)^2/(1-r) > (1+\alpha)(1-\alpha r^2)/(4-2\alpha)(1+r). \tag{2.19}$$

But $(1-\alpha r)/(1-r) \geqq 1$. Therefore (2.19) is satisfied if

$$(2.20) \qquad (4-2\alpha)(1-\alpha)r+3(1-\alpha)(1-\alpha r^2)>0.$$

(2.20) is true for $0 \leqq r < 1$. (2.18) at $b=-1$ becomes

$$(2.21) \qquad (1+\alpha r)^2/(1+r)<(1+\alpha)(1-\alpha r^2)/(4-2\alpha)(1-r).$$

(2.21) is true for $r>r(\alpha)$, where $r(\alpha)$ is given by (2.1). From (2.15) we have

$$(2.22) \qquad \frac{zF'(z)}{F(z)}=\frac{1-2\alpha bz+(2\alpha-1)z^2}{1-2bz+z^2}.$$

Therefore from (2.22) and (1.2) we get

$$f(z)=F(z)\left(\frac{1-(1+\alpha)bz+\alpha z^2}{1-2bz+z^2}\right).$$

Differentiating $f(z)$ logarithmically with respect to z, we obtain

$$\begin{aligned}\frac{zf'(z)}{f(z)}&=\frac{zF'(z)}{F(z)}+\frac{z(1-\alpha)(b-2z+bz^2)}{(1-(1+\alpha)bz+\alpha z^2)(1-2bz+z^2)}\\&=2R_0-1+(a-R_0)(1+\alpha-2R_0)/R_0(1-\alpha)\\&=(2/(1-\alpha))\{((1+\alpha)(4-2\alpha)a)^{1/2}-1-a\} \quad \text{at } z=r.\end{aligned}$$

The equality sign in the second inequality of (2.2) is attained for the function

$$(2.23) \qquad F(z)=z(1+z)^{-2+2\alpha} \quad \text{at } z=r.$$

From (2.23) and (1.2) we get $f(z)=F(z)(1+\alpha z)/(1+z)$,

$$\therefore \frac{zf'(z)}{f(z)}=\frac{1-(2-4\alpha)z+\alpha(2\alpha-1)z^2}{(1+z)(1+\alpha z)}.$$

Hence the proof of the lemma is complete.

3. Statement and proof of the main result.

THEOREM. *Let $f(z)$ be given by* (1.2) *and let $r(\alpha)$ be the root, unique in* $(\frac{1}{2}, 1]$, *of the equation* (2.1). *Then $f(z)$ is starlike of order β for $|z|<r_0$, where r_0 is the smallest positive root of the equation*

$$(3.1) \qquad 1-\beta+(2(2\alpha-1)-\beta(1+\alpha))r+\alpha(2\alpha-\beta-1)r^2=0 \qquad \textit{if } 0 \leqq r_0 \leqq r(\alpha),$$

and of the equation

$$a^2 + \{\beta(1 - \alpha) - 2(1 + \alpha - \alpha^2)\}a + \{1 + \beta(1 - \alpha)/2\}^2 = 0 \qquad \text{if } r(\alpha) \leqq r_0 < 1, \tag{3.2}$$

where $a=(1-\alpha r^2)/(1-r^2)$. *This result is sharp.*

PROOF. Since $f(z)$ is as stated in the Lemma,

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} &\geqq \frac{1 - (2 - 4\alpha)r + \alpha(2\alpha - 1)r^2}{(1 + r)(1 + \alpha r)} - \beta \\ &= \frac{1 - \beta + (2(2\alpha - 1) - \beta(1 + \alpha))r + \alpha(2\alpha - \beta - 1)r^2}{(1 + r)(1 + \alpha r)} \\ &\qquad \text{if } 0 \leqq r \leqq r(\alpha),\end{aligned} \tag{3.3}$$

and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geqq \frac{2}{1 - \alpha}\left\{((1 + \alpha)(4 - 2\alpha)a)^{1/2} - 1 - a - \frac{\beta(1 - \alpha)}{2}\right. \qquad \text{if } r(\alpha) \leqq r < 1. \tag{3.4}$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geqq 0 \qquad \text{if } (1 - \beta) + (2(2\alpha - 1) - \beta(1 + \alpha))r + \alpha(2\alpha - \beta - 1)r^2 \geqq 0, \tag{3.5}$$

and

$$a^2 + \{\beta(1 - \alpha) - 2(1 + \alpha - \alpha^2)\}a + \{1 + \beta(1 - \alpha)/2\}^2 \geqq 0. \tag{3.6}$$

(3.5) is valid only when $0 \leqq r \leqq r_0 \leqq r(\alpha)$, and (3.6) is valid only when $r(\alpha) \leqq r \leqq r_0 < 1$.

The equality in (3.3) is attained for the function given by (2.23), and that in (3.4) for the function given by (2.15).

By taking $\beta=0$ in (3.1) and (3.2), we obtain Theorem 3.2 of V. Singh and R. M. Goel [**5**] as a corollary of the above theorem.

One can easily see that Theorem 1 of Libera and Livingston [**1**] follows from (3.1) which they obtain under the restriction $\alpha \leqq \beta < 1$. Thus, if $\beta=0$, α has to be zero. In our case if $\beta=0$, α need not be zero.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, KANPUR-208016, INDIA