

ON POLYNOMIAL DENSITY IN $A_q(D)$

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ABSTRACT. Let D be a bounded Jordan domain. Define $A_q(D)$, the Bers space, to be the Banach space of holomorphic functions on D , such that $\iint_D |f| \lambda_D^{2-q} dx dy$ is finite, where $\lambda_D(z)$ is the Poincaré metric for D . It is well known that the polynomials are dense in $A_q(D)$ for $2 \leq q < \infty$ and we shall prove they are dense in $A_q(D)$ for $1 < q < 2$ if the boundary of D is rectifiable. Also some remarks are made in case the boundary of D is not rectifiable.

1. Statement of the problem. Let D be a bounded Jordan domain and define $A_q(D)$, $1 < q < \infty$, to be the Banach space of holomorphic functions on D , such that,

$$(1.1) \quad \|f\|_q = \iint_D |f(z)| \lambda_D^{2-q}(z) dx dy < \infty,$$

where $\lambda_D(z)$ is the Poincaré metric for D . If ψ is a Riemann mapping function from D onto U , the unit disk, and $\phi = \psi^{-1}$, then

$$(1.2) \quad \iint_D \lambda_D^{2-q} dx dy = \iint_U |\phi'(z)|^q (1 - |z|^2)^{q-2} dx dy.$$

We note that the boundary of D is rectifiable if and only if $\phi' \in H^1(U)$, the Hardy class, and hence by a theorem of Carleson (cf. [4]), (1.2) is finite for all $q > 1$. Since D is bounded it follows immediately that the polynomials belong to $A_q(D)$ in this case.

The question of polynomial density in $A_q(D)$ has been considered by various authors; Bers [3] and Knopp [7] proved that the polynomials were dense in $A_q(D)$, for $q \geq 2$, without any assumption on the mapping functions ψ and ϕ . Later Sheingorn [11] proved that the polynomials are dense in $A_q(U^*)$, $1 < q < \infty$, where U^* is a particular Jordan region which was first introduced by Earle and Marden [5] and arises again in the proof of Knopp's Main Lemma (cf. [7]). Next Metzger and Sheingorn [9] showed that if $\phi' \in H^p(U)$, for some $p > 1$, or if D is a Smirnov domain,

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then the polynomials are dense in $A_q(D)$, $1 < q < \infty$. Finally the author recently proved [10] that if $\phi' \in H^1(U)$ and $q > 3/2$ the polynomials are dense in $A_q(D)$. We now complete the partial result of [10] by proving

THEOREM 1. *Let D be a bounded Jordan domain with rectifiable boundary. Then the polynomials are dense in $A_q(D)$; $1 < q < \infty$.*

REMARK. If $q \leq 1$, then there is nothing to be proved since $A_q(D) = \{0\}$ in this case (cf. [8, Theorem 3]). Before proving the theorem we recall that if $f \in H^s(U)$ ($0 < s < \infty$) then $f = Fg$, where F is an inner function i.e. $|F(z)| \leq 1$ in U and $|F(e^{i\theta})| = 1$ a.e. on the boundary of U . The g is an outer function so that

$$g(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) dt\right\}$$

where $\gamma \in \mathbf{R}$, $\psi(t) \geq 0$, $\log \psi(t) \in L^1$ and $\psi(t) \in L^s$, (cf. Duren [4] for a complete exposition).

2. Proof of Theorem 1. Clearly without loss of generality we can assume that $D \subset U$ and $\phi(0) = 0$. Via Sheingorn [11] it suffices to show that there exist polynomials $P_n(z)$ such that

$$(2.1) \quad \lim_{n \rightarrow \infty} \|1 - P_n \phi'^q\|_q = 0.$$

To see that (2.1) holds, we note that Gamelin [6, Theorem 2] proved that if $f \in H^s(U)$ for any $s \in (0, 1)$ there exists an inner function F and $g \in H^{ns}(U)$ ($ns \geq 1$), an outer function, such that $f = Fg^n$. Since $\phi'^q \in H^{1/q}(U)$ it follows $\phi'^q = Fg^n$. Since $F(z) = \phi'^q(z)/g^n(z)$ it follows that $|F(z)| \geq \delta(1 - |z|^2)^c$, for some positive δ and c . This follows since $|\phi'^q(z)| \geq M_1(1 - |z|^2)^q$, because ϕ' is the derivative of a bounded schlicht function and $|g^n(z)| \leq M_2(1 - |z|^2)^{-q}$ because $g \in H^{n/q}(U)$. Now one need only take $\delta = M_1/M_2$ and $c = 2q$. Hence for any $q > 1$

$$(2.2) \quad \iint_U |F(z)| (1 - |z|^2)^{q-2} m^{-\epsilon}(r) dx dy < \infty,$$

when $m(r) = \min_{\theta} |F(re^{i\theta})|$. Hence by [11], given $\epsilon_0 > 0$, there exists a polynomial $Q(z)$, such that

$$(2.3) \quad \iint_U |1 - QF| (1 - |z|^2)^{q-2} dx dy < \epsilon_0.$$

Now g is an outer function in $H^{n/q}(U)$, with $n/q \geq 1$, and Gamelin (cf. [6] and in particular the proof of Theorem 2) proved that given an $\epsilon_1 > 0$

there exists a polynomial $P(z)$ such that

$$(2.4) \quad \left(\int_0^{2\pi} |1 - P(re^{i\theta})g^n(re^{i\theta})|^{1/q} d\theta \right)^q < \varepsilon_1,$$

where ε_1 is independent of r . Since $\|f\|_q \leq K \left(\int_0^{2\pi} |f(re^{i\theta})|^{1/q} d\theta \right)^q$ for all $q > 1$ by Duren's extension of Carleson's result (cf. [4]), it follows that

$$(2.5) \quad \iint_U |1 - Pg^n| (1 - |z|^2)^{q-2} dx dy < K\varepsilon_1.$$

Now a simple argument completes the proof, since given any $\varepsilon > 0$,

$$\begin{aligned} & \iint_U |1 - PQ\phi'^q| (1 - |z|^2)^{q-2} dx dy \\ & \leq \iint_U |1 - QF| (1 - |z|^2)^{q-2} dx dy \\ & \quad + \iint_U |QF - PQFg^n| (1 - |z|^2)^{q-2} dx dy \\ & < \varepsilon_0 + \varepsilon_1 \sup_U |Q(z)| < \varepsilon \end{aligned}$$

for suitably chosen ε_0 and ε_1 . Thus (2.1) holds and the proof is complete.

REMARK. If one lets G be a Fuchsian group acting on D , then as a corollary to Theorem 1, we get that the Poincaré series of polynomials are dense in $A_q(D, G)$ (cf. Bers [2] and Knopp [7] for the appropriate formulation).

3. On nonrectifiable boundaries. If $\phi' \notin H^1(U)$, one can still obtain some results using (2.2) if (1.2) is finite. In fact we have

THEOREM 2. *Suppose $t_0 = \inf\{t : \iint_D \lambda_D^{2-t} dx dy < \infty\} < 2$. If $\phi'(U)$ omits a set of positive (logarithmic) capacity in the plane, then the polynomials are dense in $A_q(D)$, for $t_0 < q < \infty$.*

The proof follows trivially from the fact that Aharonov, Shapiro and Shields [1] have shown that if ϕ' satisfies the condition of the theorem then

$$(3.1) \quad \iint_U |\phi'|^2 m^{-\varepsilon}(r) dx dy < \infty$$

for some $\varepsilon > 0$, where $m(r) = \inf_{\theta} |\phi'(re^{i\theta})|$. Thus we need only show

LEMMA 3. Let t_0 be as above. Suppose for some $q_0 > t_0$

$$(3.2) \quad \iint_U |\phi'|^{q_0} (1 - |z|^2)^{q_0-2} m^{-\varepsilon}(r) dx dy < \infty.$$

Then the polynomials are dense in $A_q(D)$ for $t_0 < q < \infty$.

PROOF. As before we need only show that for q such that $t_0 < q < q_0$,

$$I = \iint_U |\phi'|^q (1 - |z|^2)^{q-2} m^{-\varepsilon_1}(r) dx dy < \infty$$

for some $\varepsilon_1 > 0$. If $q \geq q_0$ the result follows trivially since $|\phi'(z)| \leq C(1 - |z|^2)^{-1}$ as the derivative of a bounded schlicht function.

If $q < q_0$, we choose t such that $t_0 < t < q < q_0$; we shall use Hölders inequality with $P = (q_0 - t)/(q_0 - q)$ so that $P/P - 1 = (q_0 - t)/(q - t)$; hence we have

$$\begin{aligned} & \iint_U |\phi'|^q (1 - |z|^2)^{q-2} m(r)^{\varepsilon(-q-t)/(q_0-t)} dx dy \\ &= \iint_U |\phi'|^{q_0(q-t)/(q_0-t)} (1 - |z|^2)^{q_0(q-t)/(q_0-t)} m(r)^{\varepsilon(-q-t)/(q_0-t)} \\ & \quad \times |\phi'|^{t(q_0-q)/(q_0-t)} (1 - |z|^2)^{t(q_0-q)/(q_0-t)} (1 - |z|^2)^{-2} dx dy \\ & \leq \left(\iint_U |\phi'|^{q_0} (1 - |z|^2)^{q_0} m(r)^{-\varepsilon} (1 - |z|^2)^{-2} dx dy \right)^{(q-t)/(q_0-t)} \\ & \quad \times \left(\iint_U |\phi'|^t (1 - |z|^2)^t (1 - |z|^2)^{-2} dx dy \right)^{(q_0-q)/(q_0-q)} \end{aligned}$$

by (3.2) and the fact $t_0 < t$. Thus I is finite for $\varepsilon_1 = \varepsilon(q-t)/(q_0-t)$ so as in Theorem 1, (2.1) holds and the Lemma follows.

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