ON POLYNOMIAL DENSITY IN $A_q(D)$

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Abstract. Let $D$ be a bounded Jordan domain. Define $A_q(D)$, the Bers space, to be the Banach space of holomorphic functions on $D$, such that $\int\int_D |\lambda_{D}^{2-q}| dx \, dy$ is finite, where $\lambda_{D}(z)$ is the Poincaré metric for $D$. It is well known that the polynomials are dense in $A_q(D)$ for $2 \leq q < \infty$ and we shall prove they are dense in $A_q(D)$ for $1 < q < 2$ if the boundary of $D$ is rectifiable. Also some remarks are made in case the boundary of $D$ is not rectifiable.

1. Statement of the problem. Let $D$ be a bounded Jordan domain and define $A_q(D)$, $1 < q < \infty$, to be the Banach space of holomorphic functions on $D$, such that,

\begin{equation}
\|f\|_q = \int\int_D |f(z)|^{\lambda_{D}^{2-q}} dx \, dy < \infty,
\end{equation}

where $\lambda_{D}(z)$ is the Poincaré metric for $D$. If $\psi$ is a Riemann mapping function from $D$ onto $U$, the unit disk, and $\phi = \psi^{-1}$, then

\begin{equation}
\int\int_D \lambda_{D}^{2-q} dx \, dy = \int\int_U |\phi'(z)|^q (1 - |z|^2)^{q-2} dx \, dy.
\end{equation}

We note that the boundary of $D$ is rectifiable if and only if $\phi' \in H^1(U)$, the Hardy class, and hence by a theorem of Carleson (cf. [4]), (1.2) is finite for all $q > 1$. Since $D$ is bounded it follows immediately that the polynomials belong to $A_q(D)$ in this case.

The question of polynomial density in $A_q(D)$ has been considered by various authors; Bers [3] and Knopp [7] proved that the polynomials were dense in $A_q(D)$, for $q \geq 2$, without any assumption on the mapping functions $\psi$ and $\phi$. Later Sheingorn [11] proved that the polynomials are dense in $A_q(U^*)$, $1 < q < \infty$, where $U^*$ is a particular Jordan region which was first introduced by Earle and Marden [5] and arises again in the proof of Knopp’s Main Lemma (cf. [7]). Next Metzger and Sheingorn [9] showed that if $\phi' \in H^p(U)$, for some $p > 1$, or if $D$ is a Smirnov domain,
then the polynomials are dense in $A_q(D)$, $1 < q < \infty$. Finally the author recently proved [10] that if $\phi' \in H^1(U)$ and $q > 3/2$ the polynomials are dense in $A_q(D)$. We now complete the partial result of [10] by proving

**Theorem 1.** Let $D$ be a bounded Jordan domain with rectifiable boundary. Then the polynomials are dense in $A_q(D); 1 < q < \infty$.

**Remark.** If $q \leq 1$, then there is nothing to be proved since $A_q(D) = \{0\}$ in this case (cf. [8, Theorem 3]). Before proving the theorem we recall that if $f \in H^s(U)$ $(0 < s < \infty)$ then $f = Fg$, where $F$ is an inner function i.e. $|F(z)| \leq 1$ in $U$ and $|F(e^{i\theta})| = 1$ a.e. on the boundary of $U$. The $g$ is an outer function so that

$$g(z) = e^{iy} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) \, dt \right\}$$

where $\gamma \in \mathbb{R}$, $\psi(t) \geq 0$, $\log \psi(t) \in L^1$ and $\psi(t) \in L^s$, (cf. Duren [4] for a complete exposition).

2. **Proof of Theorem 1.** Clearly without loss of generality we can assume that $D \subset U$ and $\phi(0) = 0$. Via Sheingorn [11] it suffices to show that there exist polynomials $P_n(z)$ such that

$$\lim_{n \to \infty} \|1 - P_n \phi'^q\|_q = 0. \quad (2.1)$$

To see that (2.1) holds, we note that Gamelin [6, Theorem 2] proved that if $f \in H^s(U)$ for any $s \in (0, 1)$ there exists an inner function $F$ and $g \in H^{ns}(U) (ns \geq 1)$, an outer function, such that $f = Fg^n$. Since $\phi'^q \in H^{1/q}(U)$ it follows $\phi'^q = Fg^n$. Since $F(z) = \phi'^q(z)/g_n(z)$ it follows that $|F(z)| \geq \delta (1 - |z|^2)^c$, for some positive $\delta$ and $c$. This follows since $|\phi'^q(z)| \geq M_1(1 - |z|^2)^q$, because $\phi'$ is the derivative of a bounded schlicht function and $|g_n(z)| \leq M_2 (1 - |z|^2)^{-q}$ because $g \in H^{n/q}(U)$. Now one need only take $\delta = M_1/M_2$ and $c = 2q$. Hence for any $q > 1$

$$\int_U |F(z)| (1 - |z|^2)^{q-2} m^{-q}(r) \, dx \, dy < \infty, \quad (2.2)$$

when $m(r) = \min_\downarrow |F(re^{i\theta})|$. Hence by [11], given $\varepsilon_0 > 0$, there exists a polynomial $Q(z)$, such that

$$\int_U \left|1 - QF\right| (1 - |z|^2)^{q-2} \, dx \, dy < \varepsilon_0. \quad (2.3)$$

Now $g$ is an outer function in $H^{n/q}(U)$, with $n/q \geq 1$, and Gamelin (cf. [6] and in particular the proof of Theorem 2) proved that given an $\varepsilon_1 > 0$
there exists a polynomial \( P(z) \) such that
\[
(2.4) \quad \left( \int_0^{2\pi} |1 - P(re^{i\theta})g^n(re^{i\theta})|^{1/q} \, d\theta \right)^q < \varepsilon_1,
\]
where \( \varepsilon_1 \) is independent of \( r \). Since \( \|f\|_q \leq K(\int_0^{2\pi} |f(re^{i\theta})|^{1/q} \, d\theta)^q \) for all \( q > 1 \) by Duren's extension of Carleson's result (cf. [4]), it follows that
\[
(2.5) \quad \int_U |1 - Pg^n| (1 - |z|^2)^{q-2} \, dx \, dy < K\varepsilon_1.
\]
Now a simple argument completes the proof, since given any \( \varepsilon > 0 \),
\[
\int_U |1 - PQf^n| (1 - |z|^2)^{q-2} \, dx \, dy
\]
\[
\leq \int_U |f| (1 - |z|^2)^{q-2} \, dx \, dy
\]
\[
+ \int_U |Qf - PQf^n| (1 - |z|^2)^{q-2} \, dx \, dy
\]
\[
< \varepsilon_0 + \varepsilon_1 \sup_U |Q(z)| < \varepsilon
\]
for suitably chosen \( \varepsilon_0 \) and \( \varepsilon_1 \). Thus (2.1) holds and the proof is complete.

**Remark.** If one lets \( G \) be a Fuchsian group acting on \( D \), then as a corollary to Theorem 1, we get that the Poincaré series of polynomials are dense in \( A_q(D, G) \) (cf. Bers [2] and Knopp [7] for the appropriate formulation).

3. **On nonrectifiable boundaries.** If \( \phi \notin H^1(U) \), one can still obtain some results using (2.2) if (1.2) is finite. In fact we have

**Theorem 2.** Suppose \( t_0 = \inf\{ t : \int_D \lambda(t)^2 \, dx \, dy < \infty \} < 2 \). If \( \phi'(U) \) omits a set of positive (logarithmic) capacity in the plane, then the polynomials are dense in \( A_q(D) \), for \( t_0 < q < \infty \).

The proof follows trivially from the fact that Aharonov, Shapiro and Shields [1] have shown that if \( \phi' \) satisfies the condition of the theorem then
\[
(3.1) \quad \int_U |\phi'|^2 \, m^{-\varepsilon(r)} \, dx \, dy < \infty
\]


for some $\epsilon > 0$, where $m(r) = \inf_{|\phi'(z)|} |\phi'(re^{i\theta})|$. Thus we need only show

**Lemma 3.** Let $t_0$ be as above. Suppose for some $q_0 > t_0$

(3.2) \[
\iint_{U} |\phi'|^{q_0} (1 - |z|^2)^{q_0 - 2} m^{-\epsilon_1}(r) \, dx \, dy < \infty.
\]

Then the polynomials are dense in $A_\epsilon(D)$ for $t_0 < q < \infty$.

**Proof.** As before we need only show that for $q$ such that $t_0 < q < q_0$,

$$I = \iint_{U} |\phi'|^{q} (1 - |z|^2)^{q - 2} m^{-\epsilon_1}(r) \, dx \, dy < \infty$$

for some $\epsilon_1 > 0$. If $q \geq q_0$, the result follows trivially since $|\phi'(z)| \leq C(1 - |z|^2)^{-1}$ as the derivative of a bounded schlicht function.

If $q < q_0$, we choose $t$ such that $t_0 < t < q < q_0$; we shall use Hölder's inequality with $P = (q_0 - t)/(q_0 - q)$ so that $P/P - 1 = (q_0 - t)/(q_0 - q)$; hence we have

$$I \leq \iint_{U} |\phi'|^{q_0} (1 - |z|^2)^{q_0 - 2} m(r)^{(q_0 - q)/q_0 - t)} \, dx \, dy$$

$$= \iint_{U} |\phi'|^{q_0 (q_0 - q)/(q_0 - t)} (1 - |z|^2)^{q_0 (q_0 - q)/(q_0 - t)} m(r)^{t(q_0 - q)/(q_0 - t)}$$

$$\times |\phi'|^{t(q_0 - q)/(q_0 - t)} (1 - |z|^2)^{t(q_0 - q)/(q_0 - t)(1 - |z|^2)^{-2} \, dx \, dy}$$

$$\leq \left( \iint_{U} |\phi'|^{q_0} (1 - |z|^2)^{q_0 m(r)^{(q_0 - q)/(q_0 - t)}} \, dx \, dy \right)^{(q_0 - q)/(q_0 - t)}$$

by (3.2) and the fact $t_0 < t$. Thus $I$ is finite for $\epsilon_1 = \epsilon(q - t)/(q_0 - t)$ so as in Theorem 1, (2.1) holds and the Lemma follows.

**Bibliography**


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