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ON POLYNOMIAL DENSITY IN $A_q(D)$

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ABSTRACT. Let D be a bounded Jordan domain. Define $A_q(D)$, the Bers space, to be the Banach space of holomorphic functions on D, such that $\iint_D |f|\lambda_D^{2-q}\,dx\,dy$ is finite, where $\lambda_D(z)$ is the Poincaré metric for D. It is well known that the polynomials are dense in $A_q(D)$ for $2\leqq q<\infty$ and we shall prove they are dense in $A_q(D)$ for $1<q<2$ if the boundary of D is rectifiable. Also some remarks are made in case the boundary of D is not rectifiable.

1. **Statement of the problem.** Let D be a bounded Jordan domain and define $A_q(D)$, $1<q<\infty$, to be the Banach space of holomorphic functions on D, such that,

$$\|f\|_q = \iint_D |f(z)|\,\lambda_D^{2-q}(z)\,dx\,dy < \infty, \tag{1.1}$$

where $\lambda_D(z)$ is the Poincaré metric for D. If ψ is a Riemann mapping function from D onto U, the unit disk, and $\phi=\psi^{-1}$, then

$$\iint_D \lambda_D^{2-q}\,dx\,dy = \iint_U |\phi'(z)|^q\,(1-|z|^2)^{q-2}\,dx\,dy. \tag{1.2}$$

We note that the boundary of D is rectifiable if and only if $\phi' \in H^1(U)$, the Hardy class, and hence by a theorem of Carleson (cf. [**4**]), (1.2) is finite for all $q>1$. Since D is bounded it follows immediately that the polynomials belong to $A_q(D)$ in this case.

The question of polynomial density in $A_q(D)$ has been considered by various authors; Bers [**3**] and Knopp [**7**] proved that the polynomials were dense in $A_q(D)$, for $q\geqq 2$, without any assumption on the mapping functions ψ and ϕ. Later Sheingorn [**11**] proved that the polynomials are dense in $A_q(U^*)$, $1<q<\infty$, where U^* is a particular Jordan region which was first introduced by Earle and Marden [**5**] and arises again in the proof of Knopp's Main Lemma (cf. [**7**]). Next Metzger and Sheingorn [**9**] showed that if $\phi' \in H^p(U)$, for some $p>1$, or if D is a Smirnov domain,

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then the polynomials are dense in $A_q(D)$, $1<q<\infty$. Finally the author recently proved [10] that if $\phi' \in H^1(U)$ and $q>3/2$ the polynomials are dense in $A_q(D)$. We now complete the partial result of [10] by proving

THEOREM 1. *Let D be a bounded Jordan domain with rectifiable boundary. Then the polynomials are dense in* $A_q(D)$; $1<q<\infty$.

REMARK. If $q\leqq 1$, then there is nothing to be proved since $A_q(D)=\{0\}$ in this case (cf. [8, Theorem 3]). Before proving the theorem we recall that if $f\in H^s(U)$ $(0<s<\infty)$ then $f=Fg$, where F is an inner function i.e. $|F(z)|\leqq 1$ in U and $|F(e^{i\theta})|=1$ a.e. on the boundary of U. The g is an outer function so that

$$g(z) = e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\psi(t)\,dt\right\}$$

where $\gamma\in \boldsymbol{R}$, $\psi(t)\geqq 0$, $\log\psi(t)\in L^1$ and $\psi(t)\in L^s$, (cf. Duren [4] for a complete exposition).

2. **Proof of Theorem 1.** Clearly without loss of generality we can assume that $D\subset U$ and $\phi(0)=0$. Via Sheingorn [11] it suffices to show that there exist polynomials $P_n(z)$ such that

$$\lim_{n\to\infty}\|1 - P_n\phi'^q\|_q = 0. \tag{2.1}$$

To see that (2.1) holds, we note that Gamelin [6, Theorem 2] proved that if $f\in H^s(U)$ for any $s\in(0,1)$ there exists an inner function F and $g\in H^{ns}(U)$ $(ns\geqq 1)$, an outer function, such that $f=Fg^n$. Since $\phi'^q\in H^{1/q}(U)$ it follows $\phi'^q=Fg^n$. Since $F(z)=\phi'^{q(z)}/g_n(z)$ it follows that $|F(z)|\geqq \delta(1-|z|^2)^c$, for some positive δ and c. This follows since $|\phi'^q(z)|\geqq M_1(1-|z|^2)^q$, because ϕ' is the derivative of a bounded schlicht function and $|g^n(z)|\leqq M_2(1-|z|^2)^{-q}$ because $g\in H^{n/q}(U)$. Now one need only take $\delta=M_1/M_2$ and $c=2q$. Hence for any $q>1$

$$\iint_U |F(z)|\,(1-|z|^2)^{q-2}m^{-\varepsilon}(r)\,dx\,dy < \infty, \tag{2.2}$$

when $m(r)=\min_\theta|F(re^{i\theta})|$. Hence by [11], given $\varepsilon_0>0$, there exists a polynomial $Q(z)$, such that

$$\iint_U |1-QF|\,(1-|z|^2)^{q-2}\,dx\,dy < \varepsilon_0. \tag{2.3}$$

Now g is an outer function in $H^{n/q}(U)$, with $n/q\geqq 1$, and Gamelin (cf. [6] and in particular the proof of Theorem 2) proved that given an $\varepsilon_1>0$

there exists a polynomial $P(z)$ such that

$$\left(\int_0^{2\pi} |1 - P(re^{i\theta})g^n(re^{i\theta})|^{1/q}\, d\theta\right)^q < \varepsilon_1, \tag{2.4}$$

where ε_1 is independent of r. Since $\|f\|_q \leqq K(\int_0^{2\pi} |f(re^{i\theta})|^{1/q}\, d\theta)^q$ for all $q>1$ by Duren's extension of Carleson's result (cf. [4]), it follows that

$$\iint_U |1 - Pg^n|\, (1 - |z|^2)^{q-2}\, dx\, dy < K\varepsilon_1. \tag{2.5}$$

Now a simple argument completes the proof, since given any $\varepsilon>0$,

$$\begin{aligned}\iint_U &|1 - PQ\phi'^q|\, (1 - |z|^2)^{q-2}\, dx\, dy \\ &\leqq \iint_U |1 - QF|\, (1 - |z|^2)^{q-2}\, dx\, dy \\ &\quad + \iint_U |QF - PQFg^n|\, (1 - |z|^2)^{q-2}\, dx\, dy \\ &< \varepsilon_0 + \varepsilon_1 \sup_U |Q(z)| < \varepsilon\end{aligned}$$

for suitably chosen ε_0 and ε_1. Thus (2.1) holds and the proof is complete.

REMARK. If one lets G be a Fuchsian group acting on D, then as a corollary to Theorem 1, we get that the Poincaré series of polynomials are dense in $A_q(D, G)$ (cf. Bers [2] and Knopp [7] for the appropriate formulation).

3. **On nonrectifiable boundaries.** If $\phi' \notin H^1(U)$, one can still obtain some results using (2.2) if (1.2) is finite. In fact we have

THEOREM 2. *Suppose $t_0 = \inf\{t : \iint_D \lambda_D^{2-t}\, dx\, dy < \infty\} < 2$. If $\phi'(U)$ omits a set of positive (logarithmic) capacity in the plane, then the polynomials are dense in $A_q(D)$, for $t_0 < q < \infty$.*

The proof follows trivially from the fact that Aharonov, Shapiro and Shields [1] have shown that if ϕ' satisfies the condition of the theorem then

$$\iint_U |\phi'|^2\, m^{-\varepsilon}(r)\, dx\, dy < \infty \tag{3.1}$$

for some $\varepsilon>0$, where $m(r)=\inf_\theta|\phi'(re^{i\theta})|$. Thus we need only show

LEMMA 3. *Let t_0 be as above. Suppose for some $q_0>t_0$*

$$\iint_U |\phi'|^{q_0}(1-|z|^2)^{q_0-2}m^{-\varepsilon}(r)\,dx\,dy<\infty. \tag{3.2}$$

Then the polynomials are dense in $A_q(D)$ for $t_0<q<\infty$.

PROOF. As before we need only show that for q such that $t_0<q<q_0$,

$$I=\iint_U |\phi'|^{q}(1-|z|^2)^{q-2}m^{-\varepsilon_1}(r)\,dx\,dy<\infty$$

for some $\varepsilon_1>0$. If $q\geqq q_0$ the result follows trivially since $|\phi'(z)|\leqq C(1-|z|^2)^{-1}$ as the derivative of a bounded schlicht function.

If $q<q_0$, we choose t such that $t_0<t<q<q_0$; we shall use Hölders inequality with $P=(q_0-t)/(q_0-q)$ so that $P/P-1=(q_0-t)/(q-t)$; hence we have

$$\begin{aligned}
&\iint_U |\phi'|^{q}(1-|z|^2)^{q-2}m(r)^{\varepsilon(-q-t)/(q_0-t)}\,dx\,dy\\
&=\iint_U |\phi'|^{q_0(q-t)/(q_0-t)}(1-|z|^2)^{q_0(q-t)/(q_0-t)}m(r)^{\varepsilon(-q-t)/(q_0-t)}\\
&\qquad\times|\phi'|^{t(q_0-q)/(q_0-t)}(1-|z|^2)^{t(q_0-q)/(q_0-t)}(1-|z|^2)^{-2}\,dx\,dy\\
&\leqq\left(\iint_U|\phi'|^{q_0}(1-|z|^2)^{q_0}m(r)^{-\varepsilon}(1-|z|^2)^{-2}\,dx\,dy\right)^{(q-t)/(q_0-t)}\\
&\qquad\times\left(\iint_U|\phi'|^{t}(1-|z|^2)^{t}(1-|z|^2)^{-2}\,dx\,dy\right)^{(q_0-q)/(q_0-q)}
\end{aligned}$$

by (3.2) and the fact $t_0<t$. Thus I is finite for $\varepsilon_1=\varepsilon(q-t)/(q_0-t)$ so as in Theorem 1, (2.1) holds and the Lemma follows.

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