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THE EXCESS OF SETS OF COMPLEX EXPONENTIALS

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ABSTRACT. Let $\Lambda=\{\lambda_n\}$ be a complex sequence and denote its associated set of complex exponentials $\{\exp(i\lambda_n x)\}$ by $e(\Lambda)$. Redheffer and Alexander have shown that if $\sum|\lambda_n-\mu_n|<\infty$ then $e(\Lambda)$ and $e(\mu)$ have the same excess over their common completeness interval. This paper shows this result to be the best possible.

1. **Introduction.** Let $\Lambda=\{\lambda_n\}$ be a complex sequence and denote its associated set of complex exponentials $\{\exp(i\lambda_n x)\}$ by $e(\Lambda)$. The properties of $e(\Lambda)$ can often be predicted from analyzing the distribution of Λ, for instance, its completeness interval **[2]**, **[6]**, convergence rates **[7]**, and norm inequalities **[4]**, **[5]**, **[8]**. In this paper, a condition derived by Redheffer and Alexander **[1]** which is sufficient for preserving the excess of a set is shown to be the best possible.

Let $\Lambda=\{\lambda_n\}$ be a complex sequence; $e(\Lambda)$ is complete in $L^2(-a, a)$ if the following condition is satisfied: if $f\in L^2(-a, a)$ and

$$\int_{-a}^{a} f(x)\exp(i\lambda_n x)\,dx = 0$$

for each n, then $f\equiv 0$. The interval I is the completeness interval for $e(\Lambda)$ if the set is complete on all shorter intervals but on no longer intervals. $e(\Lambda)$ has excess $E(\Lambda)$ on an interval if it remains complete when E terms are removed but not when $E+1$ terms are removed. The range of E may include negative integers as well as $\pm\infty$ by analogous definitions. The term excess is well defined provided Λ satisfies, $n\neq m$ implies $\lambda_n\neq\lambda_m$, and this condition is implicit throughout. We have $E=+\infty$ on intervals shorter than I and $E=-\infty$ on intervals longer than I and so $E(\Lambda)$ will always refer to I, the only interval of interest. With complete generality set $I=[-\pi, \pi]$.

A sequence Λ is canonically indexed if $0\leqq n<m$ implies that $|\lambda_n|\leqq|\lambda_m|$ and $|\lambda_{-n}|\leqq|\lambda_{-m}|$ and is regular if $\inf_{n\neq m}\{|\lambda_n-\lambda_m|\}>0$.

2. **Statement of results.** Let $w(n)$ be a positive weight function defined on the integers and Λ and $U=\{\mu_n\}$ be complex sequences.

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THEOREM I (REDHEFFER-ALEXANDER [1]). *If* $w(n)\geqq\delta>0$, *for all* n, *then* $\sum|\lambda_n-\mu_n|w(n)<\infty$ *implies* $E(\Lambda)=E(U)$.

THEOREM II. *If* $\inf\{w(n)\}=0$, *then there exist real, regular, sequences* Λ *and* U *such that* $\sum|\lambda_n-\mu_n|w(n)<\infty$, *when* Λ *is canonically indexed, but* $-\infty<E(\Lambda)<E(U)<\infty$. *Theorems* I *and* II *characterize the weight functions* $w(n)$ *with the property*: $\sum|\lambda_n-\mu_n|w(n)<\infty$ *implies* $E(\Lambda)=E(U)$, *as those which satisfy* $\inf\{w(n)\}>0$.

Theorem II remains valid when $w(n)$ is allowed to take on the value $+\infty$, provided $0\cdot\infty=0$, and thus no gap theorem can remove the restriction $\inf\{w(n)\}>0$.

3. **Proof of Theorem II.** We may suppose that $\inf_{n>0}\{w(n)\}=0$ and let n_j be the first integer for which $w(n)\leqq j^{-3}$, $j=1,2,\cdots$. Define a sequence of positive integers $\{k_j\}$ by: k_0=arbitrary, large integer and for $j=1,2,\cdots$, $k_j=\inf\{n|n=n_{j+m}$ for some $m\geqq 0$ and $n\geqq k_{j-1}^2\}$. Theorem II follows from Lemma 3 and Lemma 4. The direct product definition $F(z)=\prod(1-z/\lambda_n)$ will be used to designate $F(z)=\lim_{R\to\infty}\prod_{|\lambda_n|<R}(1-z/\lambda_n)$ with convergence easily verified if there is no justification given.

LEMMA 3. *There is a real, even, regular sequence* Λ *satisfying* $\{\pm k_j, j\neq 0\}\subset\Lambda$, $0\notin\Lambda$, *and* $-\infty<E(\Lambda)<0$, *such that if* $Q(z)=\prod(1-z/\lambda_n)$ *then* Q *is of exponential type* π, $Q(x)\in L^2(-\infty,\infty)$, *and*

$$\sum j^2\int_{k_{j+1}}^{k_{j+2}}|Q(x)|^2\,dx=\infty.$$

LEMMA 4 (REDHEFFER-ALEXANDER [1]). *Let* Λ *and* U *be real sequences with* $0\notin\Lambda$, U *and* $n_\lambda(r)$ $(n_\mu(r))$ *denote the number of terms* λ_n (*resp.* μ_n) *in the interval* $(0,r)$, *counted negatively for negative* r, *and set* $\Delta r=n_\lambda(r)-n_\mu(r)$. *If* $|\Delta r|\leqq H$ *eventually, then* $|E(\Lambda)-E(U)|\leqq 4H+2$.

Suppose Λ and $Q(z)$ are as in Lemma 3 and define the sequence U by:

$$\begin{aligned}\mu_n&=\lambda_n && \text{if } \lambda_n\neq k_j,\\ &=k_k-l_j && \text{if } \lambda_n=k_j \text{ for some } j,\end{aligned}$$

where l_j is selected so that $|l_j-j|\leqq 1$ and the sequence U remains regular. Lemma 4 holds with $H=1$ so that $|E(\Lambda)-E(U)|\leqq 6$ and since $w(n_{j+m})\leqq j^{-3}$, we have $\sum|\lambda_n-\mu_n|w(n)\leqq\sum(j+1)j^{-3}<\infty$.

Set $P(z)=\prod(1-z/\mu_n)$ and $R(z)=P(z)/Q(z)$ so that

$$R(z)=\prod(1-l_jz/(k_j-z)(k_j-l_j)).$$

For a fixed z, let k_m denote one of the closest k_j to z and set

$$f(z)=|z|\sum_{j\neq m}l_j/|k_j-z|\,(k_j-l_j). \tag{1}$$

If $|\arg z|\geqq\eta>0$, then $|k_j-z|\geqq|z|\sin\eta$, and since $k_j\geqq k_{j-1}^2$ for all j, for $|\arg z|<\eta$ there is a constant A not depending on z so that $A|k_j-z|\geqq|z|$ for $j\neq m$. From the convergence of $\sum l_j|k_j-l_j|_{-1}$, it follows that the series in (1) converges uniformly and that $f(z)$ is uniformly bounded in the complex plane. Thus there is a constant A' so that

$$A' \leqq |R(z)|\,|k_m - z|/|k_m - l_m - z| \leqq A'^{-1}.$$

Hence, $P(z)$ has exponential type π and

$$\begin{aligned}\int |P(x)|^2\,dx &\geqq A' \sum \int_{k_j+1}^{k_j+2} \left(Q(x)\,\frac{(k_j - l_j - x)}{(k_j - x)}\right)^2 dx \\ &\geqq A' \sum \left(\frac{j}{2}\right)^2 \int_{k_j+1}^{k_j+2} |Q(x)|^2\,dx = \infty.\end{aligned}$$

It must be that $E(U)\geqq 0$, for if $-\infty<E(U)<0$, then by the Paley-Wiener theorem, there would be an entire function $F(z)$ of exponential type π which satisfies $F(x)\in L^2(-\infty,\infty)$ and whose zero set is $U\cup\{z_j\}$, $j=1,2,\cdots,-E(U)-1$, for some nonzero complex numbers z_j. By a theorem of Lindelöf, $F(z)=a\exp(bz)P(z)\pi(1-z/z_j)$ for constants a and b. An examination of the indicator functions for F and P shows that $b=0$ and thus $F(x)\notin L^2(-\infty,\infty)$, a contradiction.

4. **Proof of Lemma 3.** The sequence Λ is constructed from the integers so that $Q(z)$ behaves like $\sin\pi z/\pi z$ except in neighborhoods of the set $\{\pm k_j\}$, $j=1,2,\cdots$, where $|Q(x)|$ assumes relatively large values.

Let $\{l_j\}$ be a sequence of positive integers satisfying $l_j\leqq k_j^\alpha$ for some α, $0<\alpha<1$, and m be a fixed positive integer, all to be specified later. For $n>0$, set

$$\begin{aligned}\lambda_n &= n - m, && k_j - l_j \leqq n < k_j,\\ &= n - m + \tfrac{1}{2}, && k_j - l_j - m \leqq n < k_j - l_j,\\ &= n, && \text{otherwise,}\end{aligned}$$

and for $n<0$, set $\lambda_n=-\lambda_{-n}$. The sequence Λ is real, regular, and even with $0\notin\Lambda$ and $\{\pm k_j\}\subset\Lambda$. For $U=\{n\}$, $n\neq 0$, Lemma 4 holds with $m=H$ so that $E(\Lambda)$ is finite. For $j=1,2,\cdots$, define a sequence of functions $r_{\pm j}(z)$ by

$$r_{\pm j}(z) = \prod_{s=k_j-m}^{k_j-1} \left(\frac{1 \pm z(l_j - \frac{1}{2})}{\lambda_{s-l_s}(s - z)}\right). \tag{2}$$

For $Q(z)=\pi(1-z/\lambda_n)$, we then have

$$\pi z Q(z) = \sin \pi z\left(\lim_{J\to\infty} \prod_{j<J} r_j(z)\right). \tag{3}$$

The influence of the term $r_j(z)$ is only local since there is a constant A so that uniformly in z and $j>0$, we obtain $|\ln|r_j(z)|| \leqq Al_j/k_j$ whenever $|z-k_j| \geqq k_j/2$. We can assume that $k_j \geqq 2^j$ and obtain

$$(4) \qquad \left|\ln \prod{}' | r_j(z)|\right| \leqq 2A \sum 2^{(\alpha-1)j}, \qquad \alpha - 1 < 0,$$

where the $'$ denotes deletion of those terms for which $|z-k_j|<k_j/2$. From (4) it is clear that $Q(z)$ has exponential type π.

For any real x satisfying $|x-k_j| \leqq k_j/2$, the absolute value of each term in (2), $s=k_j-m, \cdots, k_j-1$, is dominated by $\max\{2, 3xl_j/\lambda_{s-l_j}|s-x|\}$. This bound and the inequality $|\sin \pi x| \leqq \pi|s-x|$ applied to (2), (3) and (4) show that there is a constant B independent of j so that

$$(5) \qquad \int_{|x-k_j| \leqq 2m} |Q(x)|^2 \, dx \leqq Bml_j^{2m}/k_j^2,$$

and

$$(6) \qquad \int_{2m \leqq |x-k_j| \leqq k_j/2} |Q(x)|^2 \, dx \leqq B \int_{|x-k_j| \leqq kj/2} x^{-2} \, dx + \frac{Bl_j^{2m}}{k_j^2} \int_{|x-k_j| \geqq m} |x - k_j|^{-2m} \, dx.$$

Thus, $Q(x) \in L^2(-\infty, \infty)$ if $\sum l_j^{2m}/k_j^2 < \infty$.

Similarly, $\int_{k_j+1}^{k_j+2} |Q(x)|^2 \, dx \geqq B' l_j^{2m}/k_j^2$ uniformly in j for some nonzero B'. It is clear that by selecting $m=2$ and $l_j = [(k_j/j)^{1/2}]$, Lemma 3 is satisfied.

5. **Remarks and extensions.** Without further restrictions on $w(n)$, the conclusion of Theorem II cannot be altered to give $|E(\Lambda)-E(U)| = \infty$. For instance, setting $w(n)=j^{-1}$ when $n=2^j=n_j$ and $w(n)=1$ otherwise, then $\sum |\lambda_n-\mu_n| w(n) < \infty$ implies $|E(\Lambda)-E(U)|<\infty$. The convention $|\infty-\infty|=0$ is used when $E(\Lambda)=\pm\infty$.

To see this, suppose that $E(\Lambda)<0$ so that there is a nontrivial $F(z)$ of exponential type π, $F(x) \in L^2(-\infty, \infty)$, whose zero set contains Λ. Without altering the conclusion $|E(\Lambda)-E(U)|<\infty$ we may assume $\lambda_n = \mu_n$ when $n \neq n_j$, $|\mu_{n_j} - \lambda_{n_j}| \leqq j$, and $|I_m \mu_n|, |I_m \lambda_n| \geqq 1$, for all n (see Elsner [3]). Let

$$Q(z) = F(z) \prod_j \frac{(1 - z/\mu_{nj})}{(1 - z/\lambda_{n_j})}$$

and suppose that λ_{n_m} is one of the closest λ_{n_j} to z. By the estimates in the proof of Theorem II,

$$A \leqq |F(z)| \, |\mu_{n_m} - z| / |Q(z)| \, |\lambda_{n_m} - z| \leqq A^{-1}$$

for a constant A. Therefore, Q has exponential type π and $Q(x)/(1+|x|) \in L^2(-\infty, \infty)$ so that $E(U)<\infty$. By symmetry, $|E(\Lambda)-E(U)|<\infty$.

If we consider only real sequences Λ and U, Theorem I and Theorem II can be restated in terms of the function $n_\lambda(r)$. We need only observe that for canonically indexed sequences Λ and U, if for some m, $\int |n_\mu(r)-n_\mu(r)+m|\,dr<\infty$, then $\sum |\lambda_n-\mu_{n+m}|<\infty$.

THEOREM I′. *Let $w(x)$ be a positive weight function for which* $\inf_x\{w(x)\}>0$. *For Λ and U are real sequences and for some m, $\int |n_\lambda(r)-n_\mu(r)+m|w(r)\,dr<\infty$, then $E(\Lambda)=E(U)$.*

THEOREM II′. *If* $\lim_{x\to\infty} w(x)=0$, *then there are real, regular sequences Λ and U such that $\int |n_\lambda(r)-n_\mu(r)|w(r)\,dr<\infty$ but $-\infty<E(\Lambda)<E(U)<\infty$.*

REFERENCES

1. W. O. Alexander and R. Redheffer, *The excess of sets of complex exponentials*, Duke Math. J. **34** (1967), 59–72. MR **34** #6432.

2. A. Beurling and P. Malliavin, *On the closure of characters and the zeros of entire functions*, Acta Math. **118** (1967), 79–93. MR **35** #654.

3. J. Elsner, *Zulässige Abänderungen von Exponential-systemen im $L^p(-A, A)$*, Math. Z. **120** (1971), 211–220.

4. M. I. Kadec, *The exact value of the Paley-Wiener constant*, Dokl. Akad. Nauk SSSR **155** (1964), 1253–1254=Soviet Math. Dokl. **5** (1964), 559–561. MR **28** #5289.

5. A. E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Z. **41** (1936), 367–379.

6. J. Kahane, *Travaux de Beurling et Malliavin*, Séminaire Bourbaki, 1961/1962, fasc. 1, Exposé 225, 2ième éd., Secrétariat mathématique, Paris, 1962. MR **26** #3561; errata, 30, 1203.

7. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ., vol. 26, Amer. Math. Soc., Providence, R.I., 1940. MR **2**, 180.

8. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R.I., 1934.

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