PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

THE EXCESS OF SETS OF COMPLEX EXPONENTIALS

DAVID R. PETERSON

ABSTRACT. Let $\Lambda=\{\lambda_n\}$ be a complex sequence and denote its associated set of complex exponentials $\{\exp(i\lambda_n x)\}$ by $e(\Lambda)$. Redheffer and Alexander have shown that if $\sum|\lambda_n-\mu_n|<\infty$ then $e(\Lambda)$ and $e(\mu)$ have the same excess over their common completeness interval. This paper shows this result to be the best possible.

1. **Introduction.** Let $\Lambda=\{\lambda_n\}$ be a complex sequence and denote its associated set of complex exponentials $\{\exp(i\lambda_n x)\}$ by $e(\Lambda)$. The properties of $e(\Lambda)$ can often be predicted from analyzing the distribution of Λ, for instance, its completeness interval **[2]**, **[6]**, convergence rates **[7]**, and norm inequalities **[4]**, **[5]**, **[8]**. In this paper, a condition derived by Redheffer and Alexander **[1]** which is sufficient for preserving the excess of a set is shown to be the best possible.

Let $\Lambda=\{\lambda_n\}$ be a complex sequence; $e(\Lambda)$ is complete in $L^2(-a, a)$ if the following condition is satisfied: if $f\in L^2(-a, a)$ and

$$\int_{-a}^{a} f(x)\exp(i\lambda_n x)\,dx = 0$$

for each n, then $f\equiv 0$. The interval I is the completeness interval for $e(\Lambda)$ if the set is complete on all shorter intervals but on no longer intervals. $e(\Lambda)$ has excess $E(\Lambda)$ on an interval if it remains complete when E terms are removed but not when $E+1$ terms are removed. The range of E may include negative integers as well as $\pm\infty$ by analogous definitions. The term excess is well defined provided Λ satisfies, $n\neq m$ implies $\lambda_n\neq\lambda_m$, and this condition is implicit throughout. We have $E=+\infty$ on intervals shorter than I and $E=-\infty$ on intervals longer than I and so $E(\Lambda)$ will always refer to I, the only interval of interest. With complete generality set $I=[-\pi, \pi]$.

A sequence Λ is canonically indexed if $0\leqq n<m$ implies that $|\lambda_n|\leqq|\lambda_m|$ and $|\lambda_{-n}|\leqq|\lambda_{-m}|$ and is regular if $\inf_{n\neq m}\{|\lambda_n-\lambda_m|\}>0$.

2. **Statement of results.** Let $w(n)$ be a positive weight function defined on the integers and Λ and $U=\{\mu_n\}$ be complex sequences.

Received by the editors May 8, 1973.

AMS (MOS) subject classifications (1970). Primary 42A64.

Key words and phrases. Completeness, excess.

THEOREM I (REDHEFFER-ALEXANDER [1]). *If* $w(n) \geqq \delta > 0$, *for all* n, *then* $\sum |\lambda_n - \mu_n| w(n) < \infty$ *implies* $E(\Lambda) = E(U)$.

THEOREM II. *If* $\inf\{w(n)\} = 0$, *then there exist real, regular, sequences* Λ *and* U *such that* $\sum |\lambda_n - \mu_n| w(n) < \infty$, *when* Λ *is canonically indexed, but* $-\infty < E(\Lambda) < E(U) < \infty$. *Theorems* I *and* II *characterize the weight functions* $w(n)$ *with the property*: $\sum |\lambda_n - \mu_n| w(n) < \infty$ *implies* $E(\Lambda) = E(U)$, *as those which satisfy* $\inf\{w(n)\} > 0$.

Theorem II remains valid when $w(n)$ is allowed to take on the value $+\infty$, provided $0 \cdot \infty = 0$, and thus no gap theorem can remove the restriction $\inf\{w(n)\} > 0$.

3. **Proof of Theorem II.** We may suppose that $\inf_{n>0}\{w(n)\} = 0$ and let n_j be the first integer for which $w(n) \leqq j^{-3}$, $j = 1, 2, \cdots$. Define a sequence of positive integers $\{k_j\}$ by: $k_0 =$ arbitrary, large integer and for $j = 1, 2, \cdots$, $k_j = \inf\{n \mid n = n_{j+m}$ for some $m \geqq 0$ and $n \geqq k_{j-1}^2\}$. Theorem II follows from Lemma 3 and Lemma 4. The direct product definition $F(z) = \prod (1 - z/\lambda_n)$ will be used to designate $F(z) = \lim_{R\to\infty} \prod_{|\lambda_n| < R} (1 - z/\lambda_n)$ with convergence easily verified if there is no justification given.

LEMMA 3. *There is a real, even, regular sequence* Λ *satisfying* $\{\pm k_j, j \neq 0\} \subset \Lambda$, $0 \notin \Lambda$, *and* $-\infty < E(\Lambda) < 0$, *such that if* $Q(z) = \prod (1 - z/\lambda_n)$ *then* Q *is of exponential type* π, $Q(x) \in L^2(-\infty, \infty)$, *and*

$$\sum j^2 \int_{k_{j+1}}^{k_{j+2}} |Q(x)|^2 \, dx = \infty.$$

LEMMA 4 (REDHEFFER-ALEXANDER [1]). *Let* Λ *and* U *be real sequences with* $0 \notin \Lambda$, U *and* $n_\lambda(r)$ $(n_\mu(r))$ *denote the number of terms* λ_n (*resp.* μ_n) *in the interval* $(0, r)$, *counted negatively for negative* r, *and set* $\Delta r = n_\lambda(r) - n_\mu(r)$. *If* $|\Delta r| \leqq H$ *eventually, then* $|E(\Lambda) - E(U)| \leqq 4H + 2$.

Suppose Λ and $Q(z)$ are as in Lemma 3 and define the sequence U by:

$$\begin{aligned} \mu_n &= \lambda_n & &\text{if } \lambda_n \neq k_j, \\ &= k_k - l_j & &\text{if } \lambda_n = k_j \text{ for some } j, \end{aligned}$$

where l_j is selected so that $|l_j - j| \leqq 1$ and the sequence U remains regular. Lemma 4 holds with $H = 1$ so that $|E(\Lambda) - E(U)| \leqq 6$ and since $w(n_{j+m}) \leqq j^{-3}$, we have $\sum |\lambda_n - \mu_n| w(n) \leqq \sum (j+1) j^{-3} < \infty$.

Set $P(z) = \prod (1 - z/\mu_n)$ and $R(z) = P(z)/Q(z)$ so that

$$R(z) = \prod (1 - l_j z/(k_j - z)(k_j - l_j)).$$

For a fixed z, let k_m denote one of the closest k_j to z and set

$$f(z) = |z| \sum_{j \neq m} l_j / |k_j - z| \, (k_j - l_j). \tag{1}$$

If $|\arg z| \geqq \eta > 0$, then $|k_j - z| \geqq |z| \sin \eta$, and since $k_j \geqq k_{j-1}^2$ for all j, for $|\arg z| < \eta$ there is a constant A not depending on z so that $A|k_j - z| \geqq |z|$ for $j \neq m$. From the convergence of $\sum l_j |k_j - l_j|_{-1}$, it follows that the series in (1) converges uniformly and that $f(z)$ is uniformly bounded in the complex plane. Thus there is a constant A' so that

$$A' \leqq |R(z)|\,|k_m - z|/|k_m - l_m - z| \leqq A'^{-1}.$$

Hence, $P(z)$ has exponential type π and

$$\begin{aligned}\int |P(x)|^2\,dx &\geqq A' \sum \int_{k_j+1}^{k_j+2} \left(Q(x) \frac{(k_j - l_j - x)}{(k_j - x)} \right)^2 dx \\ &\geqq A' \sum \left(\frac{j}{2}\right)^2 \int_{k_j+1}^{k_j+2} |Q(x)|^2\,dx = \infty.\end{aligned}$$

It must be that $E(U) \geqq 0$, for if $-\infty < E(U) < 0$, then by the Paley-Wiener theorem, there would be an entire function $F(z)$ of exponential type π which satisfies $F(x) \in L^2(-\infty, \infty)$ and whose zero set is $U \cup \{z_j\}, j = 1, 2, \cdots, -E(U) - 1$, for some nonzero complex numbers z_j. By a theorem of Lindelöf, $F(z) = a \exp(bz) P(z) \pi(1 - z/z_j)$ for constants a and b. An examination of the indicator functions for F and P shows that $b = 0$ and thus $F(x) \notin L^2(-\infty, \infty)$, a contradiction.

4. **Proof of Lemma 3.** The sequence Λ is constructed from the integers so that $Q(z)$ behaves like $\sin \pi z/\pi z$ except in neighborhoods of the set $\{\pm k_j\}, j = 1, 2, \cdots$, where $|Q(x)|$ assumes relatively large values.

Let $\{l_j\}$ be a sequence of positive integers satisfying $l_j \leqq k_j^\alpha$ for some α, $0 < \alpha < 1$, and m be a fixed positive integer, all to be specified later. For $n > 0$, set

$$\begin{aligned}\lambda_n &= n - m, \qquad k_j - l_j \leqq n < k_j, \\ &= n - m + \tfrac{1}{2}, \qquad k_j - l_j - m \leqq n < k_j - l_j, \\ &= n, \quad \text{otherwise},\end{aligned}$$

and for $n < 0$, set $\lambda_n = -\lambda_{-n}$. The sequence Λ is real, regular, and even with $0 \notin \Lambda$ and $\{\pm k_j\} \subset \Lambda$. For $U = \{n\}$, $n \neq 0$, Lemma 4 holds with $m = H$ so that $E(\Lambda)$ is finite. For $j = 1, 2, \cdots$, define a sequence of functions $r_{\pm j}(z)$ by

$$r_{\pm j}(z) = \prod_{s=k_j-m}^{k_j-1} \left(\frac{1 \pm z(l_j - \frac{1}{2})}{\lambda_{s-l_s}(s - z)} \right). \tag{2}$$

For $Q(z) = \pi(1 - z/\lambda_n)$, we then have

$$\pi z Q(z) = \sin \pi z \left(\lim_{J \to \infty} \prod_{j < J} r_j(z) \right). \tag{3}$$

The influence of the term $r_j(z)$ is only local since there is a constant A so that uniformly in z and $j>0$, we obtain $|\ln|r_j(z)|| \leqq Al_j/k_j$ whenever $|z-k_j| \geqq k_j/2$. We can assume that $k_j \geqq 2^j$ and obtain

$$\left|\ln \prod{}' | r_j(z)|\right| \leqq 2A \sum 2^{(\alpha-1)j}, \qquad \alpha - 1 < 0, \tag{4}$$

where the $'$ denotes deletion of those terms for which $|z-k_j|<k_j/2$. From (4) it is clear that $Q(z)$ has exponential type π.

For any real x satisfying $|x-k_j| \leqq k_j/2$, the absolute value of each term in (2), $s=k_j-m, \cdots, k_j-1$, is dominated by $\max\{2, 3xl_j/\lambda_{s-l_j}|s-x|\}$. This bound and the inequality $|\sin \pi x| \leqq \pi|s-x|$ applied to (2), (3) and (4) show that there is a constant B independent of j so that

$$\int_{|x-k_j| \leqq 2m} |Q(x)|^2\, dx \leqq Bml_j^{2m}/k_j^2, \tag{5}$$

and

$$\begin{aligned} \int_{2m \leqq |x-k_j| \leqq k_j/2} |Q(x)|^2\, dx \leqq B \int_{|x-k_j| \leqq k_j/2} x^{-2}\, dx \\ + \frac{Bl_j^{2m}}{k_j^2} \int_{|x-k_j| \geqq m} |x-k_j|^{-2m}\, dx. \end{aligned} \tag{6}$$

Thus, $Q(x) \in L^2(-\infty, \infty)$ if $\sum l_j^{2m}/k_j^2 < \infty$.

Similarly, $\int_{k_j+1}^{k_j+2} |Q(x)|^2\, dx \geqq B' l_j^{2m}/k_j^2$ uniformly in j for some nonzero B'. It is clear that by selecting $m=2$ and $l_j=[(k_j/j)^{1/2}]$, Lemma 3 is satisfied.

5. **Remarks and extensions.** Without further restrictions on $w(n)$, the conclusion of Theorem II cannot be altered to give $|E(\Lambda)-E(U)|=\infty$. For instance, setting $w(n)=j^{-1}$ when $n=2^j=n_j$ and $w(n)=1$ otherwise, then $\sum |\lambda_n-\mu_n| w(n)<\infty$ implies $|E(\Lambda)-E(U)|<\infty$. The convention $|\infty-\infty|=0$ is used when $E(\Lambda)=\pm\infty$.

To see this, suppose that $E(\Lambda)<0$ so that there is a nontrivial $F(z)$ of exponential type π, $F(x) \in L^2(-\infty,\infty)$, whose zero set contains Λ. Without altering the conclusion $|E(\Lambda)-E(U)|<\infty$ we may assume $\lambda_n=\mu_n$ when $n \neq n_j$, $|\mu_{n_j}-\lambda_{n_j}| \leqq j$, and $|I_m\mu_n|, |I_m\lambda_n| \geqq 1$, for all n (see Elsner [3]). Let

$$Q(z) = F(z) \prod_j \frac{(1 - z/\mu_{n_j})}{(1 - z/\lambda_{n_j})}$$

and suppose that λ_{n_m} is one of the closest λ_{n_j} to z. By the estimates in the proof of Theorem II,

$$A \leqq |F(z)|\,|\mu_{n_m} - z|/|Q(z)|\,|\lambda_{n_m} - z| \leqq A^{-1}$$

for a constant A. Therefore, Q has exponential type π and $Q(x)/(1+|x|) \in L^2(-\infty, \infty)$ so that $E(U)<\infty$. By symmetry, $|E(\Lambda)-E(U)|<\infty$.

If we consider only real sequences Λ and U, Theorem I and Theorem II can be restated in terms of the function $n_\lambda(r)$. We need only observe that for canonically indexed sequences Λ and U, if for some m, $\int |n_\mu(r)-n_\mu(r)+m|\, dr<\infty$, then $\sum |\lambda_n-\mu_{n+m}|<\infty$.

THEOREM I′. *Let $w(x)$ be a positive weight function for which* $\inf_x\{w(x)\}>0$. *For Λ and U are real sequences and for some m,* $\int |n_\lambda(r)-n_\mu(r)+m|w(r)\, dr<\infty$, *then* $E(\Lambda)=E(U)$.

THEOREM II′. *If* $\lim_{x\to\infty} w(x)=0$, *then there are real, regular sequences Λ and U such that* $\int |n_\lambda(r)-n_\mu(r)|w(r)\, dr<\infty$ *but* $-\infty<E(\Lambda)<E(U)<\infty$.

REFERENCES

1. W. O. Alexander and R. Redheffer, *The excess of sets of complex exponentials*, Duke Math. J. **34** (1967), 59–72. MR **34** #6432.

2. A. Beurling and P. Malliavin, *On the closure of characters and the zeros of entire functions*, Acta Math. **118** (1967), 79–93. MR **35** #654.

3. J. Elsner, *Zulässige Abänderungen von Exponential-systemen im* $L^p(-A, A)$, Math. Z. **120** (1971), 211–220.

4. M. I. Kadec, *The exact value of the Paley-Wiener constant*, Dokl. Akad. Nauk SSSR **155** (1964), 1253–1254=Soviet Math. Dokl. **5** (1964), 559–561. MR **28** #5289.

5. A. E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Z. **41** (1936), 367–379.

6. J. Kahane, *Travaux de Beurling et Malliavin*, Séminaire Bourbaki, 1961/1962, fasc. 1, Exposé 225, 2ième éd., Secrétariat mathématique, Paris, 1962. MR **26** #3561; errata, 30, 1203.

7. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ., vol. 26, Amer. Math. Soc., Providence, R.I., 1940. MR **2**, 180.

8. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R.I., 1934.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916