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APPROXIMATION OF MULTIPLIERS

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ABSTRACT. We note some necessary and sufficient conditions concerning norm approximation of Fourier multipliers, and give an example to show that $M_q(Z)$, the space of Fourier multipliers of type (q, q), is not norm dense in $M_p(Z)$ when $1 \leqq q < p \leqq 2$. An extension of this example to more general groups is indicated.

1. Let G and Γ be dual L.C.A. groups with Haar measures dx and $d\gamma$ respectively. Let $L^p(G)$, $1 \leqq p \leqq \infty$, be the usual Lebesgue space on G, $C_0(G)$ the space of continuous functions on G which vanish at infinity, $C_c(G)$ the space of continuous functions on G with compact support. The Fourier transform of a function f is denoted $\hat{f}$. Let $M_p(\Gamma)$ denote the space of (equivalence classes of) essentially bounded measurable functions ϕ on Γ which satisfy the condition

$$\left| \int_\Gamma \phi(\gamma) \hat{f}(\gamma) \hat{g}(\gamma) \, d\gamma \right| \leqq C \, \|f\|_p \, \|g\|_{p'} \tag{1.1}$$

for any f, g elements of $C_c(G)$ where $1 \leqq p \leqq \infty$ and $1/p + 1/p' = 1$. The norm of ϕ is the least admissible value of C in (1.1). It is known that

$$L^\infty(\Gamma) = M_2(\Gamma) \supseteq M_p(\Gamma) \supseteq M_q(\Gamma) \supseteq M_1(\Gamma) = B(\Gamma)$$

when $1 \leqq q \leqq p \leqq 2$; $B(\Gamma)$ is the space of Fourier transforms of complex regular Borel measures on G, with the inherited norm.

We shall make use of the fact that $M_p(\Gamma)$ is the dual of the Banach space $A_p(G)$ [**4**]. The space $A_p(G)$ consists of those functions in $C_0(G)$ which can be written in the form

$$h = \sum f_i * g_i \tag{1.2}$$

where f_i, $g_i \in C_c(G)$ and

$$\sum \|f_i\|_p \, \|g_i\|_{p'} < \infty. \tag{1.2'}$$

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The norm of h is the infimum of all sums (1.2′) over all possible representations (1.2) of h. The duality between A_p and M_p is expressed by the formula

$$\langle \phi, h\rangle = \sum \int_\Gamma \phi(\gamma)\hat{f}_i(\gamma)\hat{g}_i(\gamma)\, d\gamma$$

where $\phi \in M_p(\Gamma)$ and $h \in A_p(G)$ with representation (1.2). One can readily deduce from the inclusion results among the multiplier spaces that

$$C_0(G) = A_1(G) \supseteq A_q(G) \supseteq A_p(G) \supseteq A_2(G) = A(G)$$

when p and q are as above and $A(G)$ is the space of Fourier transforms of functions in $L^1(\Gamma)$, with the inherited norm. Recall that $A(G)$ is dense in $A_p(G)$.

Our first result is an extension and variant of a result of Edwards [**3**] and Ramirez [**6**].

PROPOSITION. *Let $1 \leqq q < p \leqq 2$ and let ϕ be an element of $M_p(\Gamma)$. Then the following statements are equivalent*:

(i) *ϕ is approximable in the M_p norm by elements of M_q.*

(ii) *If (h_n) is a sequence in A_p such that $\|h_n\| \leqq N$ and $\|h_n\|_{A_q} \to 0$, then $\langle \phi, h_n\rangle \to 0$.*

(iii) *If (h_α) is a net in A_p such that $\|h_\alpha\| \leqq N$ and $\langle \psi, h_\alpha\rangle \to 0$ for each ψ in M_q, then $\langle \phi, h_\alpha\rangle \to 0$.*

PROOF. It is simple to check that (i) implies (iii). As to the reverse implication, an application of the Hahn-Banach theorem shows that $\phi \in M_q^-$ if and only if every element λ of M_p^*, of norm 1, which annihilates M_q also annihilates ϕ. It follows from a well-known characterization of the unit ball in $M_p^* = A_p^{**}$ that such a λ is in the weak*-closure of the unit ball U_1 of A_p (injected canonically into M_p^*)—see [**1**, V.4.5]. With these observations in mind one concludes readily that (iii) implies (i).

Let X denote the normed linear space consisting of the elements of $A_p(G)$ with $\|\cdot\|_X = \|\cdot\|_{A_q}$. Then (ii) simply means that the restriction of ϕ to the sets $U_N = \{h \in X : \|h\|_{A_p} \leqq N\}$ is (norm) continuous and (iii) means that the restriction of ϕ to the sets U_N is weakly continuous. Clearly (iii) implies (ii). To prove the converse, suppose there exists a net (h_α) such that $\|h_\alpha\|_{A_p} \leqq N$, $h_\alpha \to 0$ weakly, and $\langle \phi, h_\alpha\rangle \not\to 0$. Then there exists a subnet which we shall also call (h_α) so that $\langle \phi, h_\alpha\rangle \to a \neq 0$. There exists an α_0 so that if $\varepsilon = |a|/2$, then $\langle \phi, h_\alpha\rangle \in B_\varepsilon(a) = \{z \in C : |z-a| < \varepsilon\}$, if $\alpha > \alpha_0$. Consider the set $S = \{h_\alpha\}_{\alpha > \alpha_0}$. 0 is certainly in the weak closure of S. It follows from Theorem V.3.13 of [**1**] that 0 belongs to the norm closure of co S, where co S denotes the convex hull of S. So there exists a sequence (y_n) in co S so that $\|y_n\|_{A_q} \to 0$. By condition (iii), $\langle \phi, y_n\rangle \to 0$, but $y_n \in$ co S so $\langle \phi, y_n\rangle \in$ co $B_\varepsilon(a) = B_\varepsilon(a)$. Hence $|\langle \phi, y_n\rangle| > a/2$ and $\langle \phi, y_n\rangle \not\to 0$, a contradiction.

COROLLARY. *Let ϕ be a continuous bounded function on Γ. Then the following statements are equivalent*:

(i) *ϕ is the uniform limit of Fourier-Stieltjes transforms.*

(ii) *If (h_n) is a sequence in $A(G)$ such that $\|h_n\|_A \leqq N$ and $\|h_n\|_\infty \to 0$* then $\int_\Gamma \hat{h}_n(\gamma)\phi(\gamma)\,d\gamma \to 0$.

(iii) *If (h_α) is a net in $A(G)$ such that $\|h_\alpha\|_A \leqq N$ and $\int_\Gamma \hat{h}_\alpha(\gamma)\psi(\gamma)\,d\gamma \to 0$ for each ψ in $B(\Gamma)$ then $\int_\Gamma \hat{h}_\alpha(\gamma)\phi(\gamma)\,d\gamma \to 0$.*

REMARK. The equivalence of conditions (i) and (iii) of this Corollary is the result of Edwards and Ramirez [loc. cit.].

2. **An explicit example.** Figà-Talamanca and Gaudry [**5**] have constructed an element ϕ of $M_p \cap C_0(Z)$ which is not the limit in M_p norm of elements of $A(Z)$. We show that the function ϕ constructed by them is not approximable in M_p norm by elements of $M_q(Z)$ when $1 \leqq q < p \leqq 2$. By condition (ii) of the above Proposition it suffices to produce a sequence (h_n) in $A_p(T)$ such that $\|h_n\|_{A_p} \leqq N$ for some N, $\|h_n\|_{A_q} \to 0$, and $\langle \phi, h_n \rangle \not\to 0$. Define the sequences (ρ_n) and (σ_n) of Rudin-Shapiro polynomials on T as follows:

$$\rho_0 = \sigma_0 = 1,$$
$$\rho_n(x) = \rho_{n-1}(x) + \exp(i2^{n-1}x)\sigma_{n-1}(x),$$
$$\sigma_n(x) = \rho_{n-1}(x) - \exp(i2^{n-1}x)\sigma_{n-1}(x).$$

Then

$$|\rho_n|^2 + |\sigma_n|^2 = 2(|\rho_{n-1}|^2 + |\sigma_{n-1}|^2) = \cdots = 2^{n+1}$$

from which it follows that $\|\rho_n\|_\infty \leqq 2^{(n+1)/2}$. Further, the functions $\hat{\rho}_n$ and $\hat{\sigma}_n$ take only the values 0 and ± 1 and are supported precisely on the set $\{0, \cdots, 2^n - 1\}$.

On $[2^n, 2^{n+1})$ put $\phi = (\exp(i2^n x)\rho_n(x))^\wedge \cdot 2^{-n/r}$ where $r = 2p/(2-p)$ and $n \geqq 0$. Put $\phi(k) = 0$ for $k = 0, -1, -2, \cdots$. Then $\phi \in M_p \cap C_0(Z)$—see [**5**]. Now $\rho_n = \rho_n * D_{2^n}$ where D_N denotes the Nth Dirichlet kernel. It is known [**2**, Exercise 7.5] that $\|D_N\|_p \sim B_p N^{1/p'}$ as $N \to \infty$, when $1 < p < \infty$ and B_p is a constant depending only on p. Hence

$$\|\rho_n\|_{A_p} \leqq \|\rho_n\|_{p'} \|D_{2^n}\|_p \leqq B_p 2^{(n+1)/2} 2^{n/p'} \quad \text{as } n \to \infty$$

and, if $q > 1$,

$$\|\rho_n\|_{A_q} \leqq B_q 2^{(n+1)/2} 2^{n/q'} \quad \text{as } n \to \infty.$$

Put $h_n(x) = \exp(i2^n x)\rho_n(x)/2^{(n+1)/2}2^{n/p'}$. Then $\|h_n\|_{A_p} \leqq B_p$ and $\|h_n\|_{A_q} \leqq B_q 2^{n(1/q' - 1/p')}$ as $n \to \infty$. Since $1 < q < p \leqq 2$ we have $2 \leqq p' < q' < \infty$ whence $\|h_n\|_{A_q} \to 0$ as $n \to \infty$. Finally

$$\langle \phi, h_n \rangle = \rho_n * \rho_n(0)/2^{(n+1)/2} 2^{n/q'} 2^{n/r}$$
$$= 2^n / 2^{(n+1)/2} 2^{n/p'} 2^{n/r} = 2^{-1/2}$$

which completes the proof for $q>1$. But if $\phi \in M_p \setminus M_q^-$ for $1<q<p$, then certainly $\phi \notin M_1^-$, which completes the proof.

REMARKS. (a) It is possible to generalize the argument given above to the case of an arbitrary noncompact L.C.A. group Γ, and so to produce a function ϕ in $M_p \cap C_0(\Gamma) \setminus M_q \cap C(\Gamma)^-$. An indication of how to do this, by using the structure theorem and a lifting argument, is given in [**5**].

(b) It is possible to modify the above argument to prove that there exists $\phi \in M_p \cap C_0(Z)$ which is not the limit, in M_p norm, of elements of $\bigcup_{1 \leqq q < p} M_q(Z)$. Take ϕ as above. Then it is easily seen that it suffices, given any sequence (q_n) of real numbers $1 \leqq q_n < p$ such that $q_n \to p$, to produce a sequence (h_n) in A_p such that $\|h_n\|_{A_p} \leqq N$, $\|h_n\|_{A_{q_n}} \to 0$, and $\langle \phi, h_n \rangle \not\to 0$. Define the sequence $k(n)$ so that $k(n) \to \infty$ in such a way that $k(n)(1/q_n - 1/p) \to \infty$, and put $h_n = \{\exp(i2^{k(n)}x\}\rho_{k(n)}/2^{(k(n)+1)/2}2^{k(n)/p'}$. It is easy to check that this yields the desired result.

(c) C. Fefferman (grapevine) is reported to have produced an element ϕ of $M_n \cap C_c(\boldsymbol{R}^n)$ such that $\|\phi_a - \phi\|_{M_p} \not\to 0$ as $a \to 0$, where $\phi_a(x) = \phi(x-a)$. An elementary interpolation argument shows that such a function ϕ is not approximable, in M_p norm, by elements of $M_q \cap C(\boldsymbol{R}^n)$.

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