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A NOTE ON THE SUM OF TWO CLOSED LATTICE IDEALS*

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ABSTRACT. Suppose that E is a locally convex lattice. The main results established in this note are: (a) If I, J are $\sigma(E', E)$-closed lattice ideals in the dual E' of E, then $I+J$ is $\sigma(E', E)$-closed. (b) If E is a Fréchet lattice (in particular, if E is a Banach lattice) and if I, J are closed lattice ideals in E, then $I+J$ is closed.

It is known that the sum of two closed lattice ideals in a Banach lattice is a closed lattice ideal (see Theorem 5.3 in [**1**] and Theorem 1.1 in [**2**]). In this note, we deal with the sum of two closed lattice ideals in a locally convex lattice and with the sum of the polars of two lattice ideals, that is, with the sum of two weak*-closed lattice ideals in the dual space.

A linear subspace I of a vector lattice E is a *lattice ideal* if I is solid, that is, if $x \in I$ and $|y| \leqq |x|$ imply $y \in I$. The sum of two lattice ideals in a vector lattice is a lattice ideal. A closed linear subspace I of a locally convex vector lattice E is an ideal if and only if the polar I° of I is a lattice ideal in the dual E' of E.

We refer the reader to [3] for further background information on locally convex vector lattices.

THEOREM 1. *If E is a locally convex vector lattice and if I and J are lattice ideals in E, then $(I \cap J)^\circ = I^\circ + J^\circ$.*

PROOF. It is clear that $(I \cap J)^\circ \supset I^\circ + J^\circ$. To prove the reverse inclusion, it would suffice to show that if $0 \leqq f \in (I \cap J)^\circ$, then $f \in I^\circ + J^\circ$ since $I^\circ + J^\circ$ is a lattice ideal in E'. For $x \geqq 0$ in E, define

$$\gamma(x) = \sup\{f(y) : y \in [0, x] \cap I\}.$$

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Then γ is additive and positively homogeneous on the positive cone in E; consequently, γ can be extended to a linear functional g on E (cf. proofs of V, 1.4 and V, 1.6 in [3]). Since $0 \leqq g \leqq f$ it follows that $g \in E'$. Moreover, $f-g \in I^\circ$ and $g \in J^\circ$ since $[0, x] \cap I \subset I \cap J$ for each $x \in J$. Therefore, $f=(f-g)+g \in I^\circ + J^\circ$ which completes the proof.

REMARK. The linear functional g constructed in the above proof is just the component of f in $I^{\circ\perp}$ when E' is written as the order direct sum of the bands I° and $(I^\circ)^\perp$.

COROLLARY. *If E is a locally convex vector lattice, then the sum of two $\sigma(E', E)$-closed lattice ideals in E' is $\sigma(E', E)$-closed.*[2]

PROOF. If I and J are $\sigma(E', E)$-closed lattice ideals in E', then the $\sigma(E', E)$-closure of $I+J$ is $(I^\circ \cap J^\circ)^\circ$; consequently, the conclusion follows immediately from Theorem 1.

THEOREM 2. *Suppose that I and J are lattice ideals in a locally convex vector lattice E. Then the mapping $(x, y) \rightarrow x+y$ is a weak homomorphism from $I \times J$ into E.*

PROOF. It would suffice to show that the mapping $f \rightarrow (f|_I, f|_J)$ (where $f|_I$ denotes the restriction of f to I) from E into $I' \times J'$ has a $\sigma(I' \times J', I \times J)$-closed range [3, IV 7.3]. This range is clearly contained in the $\sigma(I' \times J', I \times J)$-closed subspace $G = \{(g, h) : g \in I', h \in J', g(x) = h(x)$ for all $x \in I \cap J\}$ of $I' \times J'$. If $(g, h) \in G$, then there exist $\hat{g}$, $\hat{h}$ in E' such that $\hat{g}|_I = g$, $\hat{h}|_J = h$ (by the Hahn-Banach theorem). Since $\hat{g} - \hat{h} \in (I \cap J)^\circ$ and since $(I \cap J)^\circ = I^\circ + J^\circ$ by Theorem 1, it follows that $\hat{g} - \hat{h} = f_1 + f_2$ where $f_1 \in I^\circ$, $f_2 \in J^\circ$. But then (g, h) is the image of $f = \hat{g} - f_1 = \hat{h} + f_2$ under the mapping $f \rightarrow (f|_I, f|_J)$, that is, the range of this mapping is the $\sigma(I' \times J', I \times J)$-closed subspace G.

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[2] This Corollary was proved independently by S. Kaplan.