A NOTE ON THE SUM OF TWO CLOSED LATTICE IDEALS*

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Abstract. Suppose that $E$ is a locally convex lattice. The main results established in this note are: (a) If $I, J$ are $\sigma(E', E)$-closed lattice ideals in the dual $E'$ of $E$, then $I+J$ is $\sigma(E', E)$-closed. (b) If $E$ is a Fréchet lattice (in particular, if $E$ is a Banach lattice) and if $I, J$ are closed lattice ideals in $E$, then $I+J$ is closed.

It is known that the sum of two closed lattice ideals in a Banach lattice is a closed lattice ideal (see Theorem 5.3 in [1] and Theorem 1.1 in [2]). In this note, we deal with the sum of two closed lattice ideals in a locally convex lattice and with the sum of the polars of two lattice ideals, that is, with the sum of two weak*-closed lattice ideals in the dual space.

A linear subspace $I$ of a vector lattice $E$ is a lattice ideal if $I$ is solid, that is, if $x \in I$ and $|y| \leq |x|$ imply $y \in I$. The sum of two lattice ideals in a vector lattice is a lattice ideal. A closed linear subspace $I$ of a locally convex vector lattice $E$ is an ideal if and only if the polar $I^\circ$ of $I$ is a lattice ideal in the dual $E'$ of $E$.

We refer the reader to [3] for further background information on locally convex vector lattices.

Theorem 1. If $E$ is a locally convex vector lattice and if $I$ and $J$ are lattice ideals in $E$, then $(I \cap J)^\circ = I^\circ + J^\circ$.

Proof. It is clear that $(I \cap J)^\circ \subseteq I^\circ + J^\circ$. To prove the reverse inclusion, it would suffice to show that if $0 \leq f \in (I \cap J)^\circ$, then $f \in I^\circ + J^\circ$ since $I^\circ + J^\circ$ is a lattice ideal in $E'$. For $x \geq 0$ in $E$, define

$$
\gamma(x) = \sup \{f(y) : y \in [0, x] \cap I\}.
$$

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Then $\gamma$ is additive and positively homogeneous on the positive cone in $E$; consequently, $\gamma$ can be extended to a linear functional $g$ on $E$ (cf. proofs of V, 1.4 and V, 1.6 in [3]). Since $0 \leq g \leq f$ it follows that $g \in E'$. Moreover, $f - g \in I^0$ and $g \in J^0$ since $[0, x] \cap I \subseteq I \cap J$ for each $x \in J$. Therefore, $f = (f - g) + g \in I^0 + J^0$ which completes the proof.

Remark. The linear functional $g$ constructed in the above proof is just the component of $f$ in $I^0 \perp$ when $E'$ is written as the order direct sum of the bands $I^0$ and $(I^0)^\perp$.

Corollary. If $E$ is a locally convex vector lattice, then the sum of two $\sigma(E', E)$-closed lattice ideals in $E'$ is $\sigma(E', E)$-closed.$^2$

Proof. If $I$ and $J$ are $\sigma(E', E)$-closed lattice ideals in $E'$, then the $\sigma(E', E)$-closure of $I + J$ is $(I^0 \cap J^0)^\circ$; consequently, the conclusion follows immediately from Theorem 1.

Theorem 2. Suppose that $I$ and $J$ are lattice ideals in a locally convex vector lattice $E$. Then the mapping $(x, y) \mapsto x + y$ is a weak homomorphism from $I \times J$ into $E$.

Proof. It would suffice to show that the mapping $f \mapsto (f|_I, f|_J)$ (where $f|_I$ denotes the restriction of $f$ to $I$) from $E$ into $I' \times J'$ has a $\sigma(I' \times J', I \times J)$-closed range [3, IV 7.3]. This range is clearly contained in the $\sigma(I' \times J', I \times J)$-closed subspace $G = \{ (g, h) : g \in I', h \in J', g(x) = h(x) \text{ for all } x \in I \cap J \}$ of $I' \times J'$. If $(g, h) \in G$, then there exist $\hat{g}$, $\hat{h}$ in $E'$ such that $\hat{g}|_I = g$, $\hat{h}|_J = h$ (by the Hahn-Banach theorem). Since $\hat{g} - \hat{h} \in (I \cap J)^\circ$ and since $(I \cap J)^\circ = I^0 + J^0$ by Theorem 1, it follows that $\hat{g} - \hat{h} = f_1 + f_2$ where $f_1 \in I^0$, $f_2 \in J^0$. But then $(g, h)$ is the image of $f = \hat{g} - f_1 = \hat{h} + f_2$ under the mapping $f \mapsto (f|_I, f|_J)$, that is, the range of this mapping is the $\sigma(I' \times J', I \times J)$-closed subspace $G$.

References


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$^2$ This Corollary was proved independently by S. Kaplan.