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ON A THEOREM OF A. PEŁCZYŃSKI

JOHN L. B. GAMLEN[1]

ABSTRACT. If Y is a weakly complete Banach space, and X is a Banach space with separable dual, then every continuous linear operator from $C_X(K)$ to Y must be weakly compact. Here $C_X(K)$ denotes the space of continuous functions on the compact Hausdorff space K, having values in X.

In 1953, A. Grothendieck [5] proved that if Y is a weakly complete Banach space, then every continuous linear map from $C(K)$ to Y must be weakly compact. (Here K is an arbitrary compact Hausdorff space.) Later A. Pełczyński [7] weakened the assumption on Y, to the requirement that c_0 is not isomorphic to any subspace of Y. In 1962, Pełczyński [8, p. 645] obtained a result which implies that $C(K)$ may be replaced in the above by $C_X(K)$, the space of continuous X-valued functions on K, where X is a reflexive Banach space. (See also J. Batt and E. J. Berg [2, p. 237], where a different but related proof is given.) Necessary and sufficient conditions on X for this to still work are not known. However, in order for every continuous linear map $T: C_X(K) \to Y$ to be weakly compact for a given Y, it is obviously necessary that every continuous operator from X to Y be weakly compact. But if X has a separable dual space, this holds for weakly complete Y, since Cantor's diagonal argument allows us to extract from every bounded sequence in X a subsequence which is weakly Cauchy. This suggests our result below.

THEOREM. *If Y is a weakly complete Banach space, and X is a Banach space whose separable subspaces have separable duals, then every continuous linear operator from $C_X(K)$ to Y must be weakly compact.*

This does not quite include the result of Pełczyński because of our slightly stronger assumption that Y is weakly complete. The interesting thing is that in the above theorem we cannot replace weak completeness of Y by the assumption that Y has no subspace isomorphic to c_0. The counterexample is the well-known space J of R. C. James [6], which has

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separable bidual, but is not reflexive. (The identity map on J fails to be weakly compact, even though J contains no copy of c_0.)

Batt and Berg's proof of Pełczyński's result proceeds via weak compactness of the adjoint map. This involves a weak compactness theorem in the space of X'-valued measures, which will not work in our case, as it depends on reflexivity of X. However this approach was used in [**1**] to prove weak compactness of a map from a C^*-algebra to a Banach space containing no copy of c_0. (The details are, needless to say, quite different.)

PROOF OF THE THEOREM. We first deal with the case that K and X (hence also X') are separable. By the representation theorem of [**2**, pp. 225–228], we may represent our map $T: C_X(K) \to Y$ as an integral with respect to a measure μ taking values in the space $[X, Y]$ of bounded operators between X, Y, and having semivariation absolutely continuous with respect to some positive regular measure λ on K. Thus the adjoint T' maps Y' into a space of measures with values in X', all of which are absolutely continuous with respect to λ. For such measures the Radon-Nikodym theorem is well known to hold, so we may embed the range of T' in $L^1(X', \lambda)$, the space of λ-integrable, X'-valued functions, by the formula:

$$\langle f, T'y'\rangle = \langle Tf, y'\rangle = \int f\, d(y'\mu) = \int f(d(y'\mu)/d\lambda)\, d\lambda, \qquad \text{for } f \in C_X(K),\ y' \in Y'.$$

Associated with the element $y'\mu$ in $T'Y'$ is the function $d(y'\mu)/d\lambda$ in $L^1(X', \lambda)$, and it is easy to see that the embedding of $T'Y'$ into L^1 is norm increasing. Since $L^1(X', \lambda)$ is separable, we conclude that $T'Y'$ is a separable subspace of the dual of $C_X(K)$. This fact enables us to use Cantor's diagonal argument to extract a subsequence $\{f_{n_m}\}$ such that for every y' in Y' the sequence $\{\langle f_{n_m}, T'y'\rangle\} = \{\langle Tf_{n_m}, y'\rangle\}$ is Cauchy. Thus by weak completeness of Y, $\{Tf_{n_m}\}$ converges weakly in Y, proving T is weakly compact.

The reduction to the case X, K, are separable is standard; see for example [**2**].

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DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA