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TOPOLOGICAL SPACES IN WHICH BLUMBERG'S THEOREM HOLDS

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ABSTRACT. H. Blumberg proved that, if f is a real-valued function defined on the real line R, then there is a dense subset D of R such that $f|D$ is continuous. J. C. Bradford and C. Goffman showed [3] that this theorem holds for a metric space X if and only if X is a Baire space. In the present paper, we show that Blumberg's theorem holds for a topological space X having a σ-disjoint pseudo-base if and only if X is a Baire space. Then we identify some classes of topological spaces which have σ-disjoint pseudo-bases. Also, we show that a certain class of locally compact, Hausdorff spaces satisfies Blumberg's theorem. Finally, we describe two Baire spaces for which Blumberg's theorem does not hold. One is completely regular, Hausdorff, cocompact, strongly α-favorable, and pseudo-complete; the other is regular and hereditarily Lindelöf.

1. In [3], J. C. Bradford and C. Goffman proved the following statement.

1.1. THEOREM. *A metric space X is a Baire space if and only if the following statement, called Blumberg's theorem, holds.*

1.2. *If f is a real-valued function defined on X, then there is a dense subset D of X such that $f|D$ is continuous.*

It is clear from the proof of 1.1 given in [3], that any topological space for which 1.2 holds is a Baire space. The purposes of this note are to show that 1.2 holds for certain classes of topological Baire spaces, and to give an example which shows that 1.2 does not hold for all completely regular, Hausdorff, Baire spaces.

1.3. LEMMA. *Suppose X and Y are topological spaces and $f: X \to Y$. Suppose that for each nonempty open subset U of X, there is a subset $K(U)$ of U such that $f|K(U)$ is continuous and $K(U)$ is not nowhere dense. Then there is a dense subset D of X such that $f|D$ is continuous.*

The proof of 1.3 is simple and is omitted.

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Part of the following proposition is implicit in the proof of 1.1 that is given in [3].

1.4. PROPOSITION. *Suppose X is a topological space and $\mathfrak{m}$ is an infinite cardinal number. The following statements are equivalent.*

(1) *If $(G_\alpha)_{\alpha\in\Gamma}$ is a family of dense open subsets of X and $|\Gamma|\leqq\mathfrak{m}$, then $\bigcap\{G_\alpha:\alpha\in\Gamma\}$ is dense in X.*

(2) *If $(K_\alpha)_{\alpha\in\Gamma}$ is a family of nowhere dense sets and $|\Gamma|\leqq\mathfrak{m}$, then $\bigcup\{K_\alpha:\alpha\in\Gamma\}$ is not open.*

(3) *If Y is a topological space of cardinality $\leqq\mathfrak{m}$ and $f:X\to Y$, then there is a dense subset D of X such that $f|D$ is continuous.*

(4) *If (Y, d) is a metric space of weight $\leqq\mathfrak{m}$ and $f:X\to Y$ then, for every $\varepsilon>0$, there is a dense subset $D(\varepsilon)$ such that $f|D(\varepsilon)$ has oscillation $\leqq\varepsilon$ at every point in $D(\varepsilon)$.*

PROOF. It is known that (1) and (2) are equivalent.

(2) implies (3). Suppose Y and f are as in (3). If U is a nonempty subset of X, then $U=\bigcup\{U\cap f^{-1}(y):y\in Y\}$. Since (2) holds, there is $y(U)$ in Y such that $U\cap f^{-1}(y(U))=K(U)$ is not nowhere dense. By 1.3, (3) holds.

(3) implies (4). Suppose (Y, d) and f are as in (4). Let $\varepsilon>0$. Let $\mathscr{B}$ be a base for Y of cardinality $\leqq\mathfrak{m}$ such that $\operatorname{dia}(B)\leqq\varepsilon$ for every B in $\mathscr{B}$. Endow $\mathscr{B}$ with the discrete topology. Define $\varphi:X\to\mathscr{B}$ so that $f(x)\in\varphi(x)$ for all x in X. Since (3) holds, there is a dense set $D(\varepsilon)$ such that $\varphi|D(\varepsilon)$ is continuous. Then $D(\varepsilon)$ is the required set.

(4) implies (2). Suppose $(K_\alpha)_{\alpha\in\Gamma}$ is as in (2). We may assume that $K_\alpha\cap K_\gamma=\varnothing$ if $\alpha\neq\gamma$. Let $Y=\Gamma\cup\{\Gamma\}$ and let d denote the zero-one metric on Y. Define $f:X\to Y$ be letting $f(x)=\alpha$ for x in K_α and $f(x)=\Gamma$ for x not in $\bigcup\{K_\alpha:\alpha\in\Gamma\}$. Since (4) holds, there is a dense subset D such that the oscillation of $f|D$ at every point of D is $\leqq\frac{1}{2}$. Then $f|D$ is continuous. And, for α in Γ, $D\cap K_\alpha=\varnothing$ since K_α is nowhere dense. Hence $\bigcup\{K_\alpha:\alpha\in\Gamma\}$ is not open.

1.5. COROLLARY. *The following statements are equivalent for a topological space X.*

(1) *X is a Baire space.*

(2) *If Y is a countable topological space and $f:X\to Y$, then there is a dense set D such that $f|D$ is continuous.*

(3) *If (Y, d) is a separable metric space and $f:X\to Y$ then, for every $\varepsilon>0$, there is a dense subset $D(\varepsilon)$ such that $f|D(\varepsilon)$ has oscillation $\leqq\varepsilon$ at every point of $D(\varepsilon)$.*

1.6. DEFINITIONS. A pseudo-base [9] for a topological space $(X,\mathscr{T})$ is a subset $\mathscr{P}$ of $\mathscr{T}$ such that every nonempty element of $\mathscr{T}$ contains a nonempty element of $\mathscr{P}$. A subfamily $\mathscr{P}$ of $\mathscr{T}$ is called σ-disjoint if $\mathscr{P}=\bigcup\{\mathscr{P}_n : n \in N\}$, where each $\mathscr{P}_n$ is a disjoint family. (Here N denotes the set of natural numbers.)

1.7. PROPOSITION. *If the Baire space* $(X, \mathscr{T})$ *has a σ-disjoint pseudo-base* $\mathscr{P}$, *then* 1.2 *holds for* X.

PROOF. Suppose $\mathscr{P}=\bigcup\{\mathscr{P}_n : n \in N\}$, where each $\mathscr{P}_n$ is a disjoint family. We may assume that, for each n in N, $G_n=\bigcup \mathscr{P}_n$ is dense in X and $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$. Let $Y=\bigcap\{G_n : n \in N\}$. Since X is a Baire space, Y is dense in X. Let $\mathscr{P}(Y)=\{P \cap Y : P \in \mathscr{P}\}$. Then $\mathscr{P}(Y)$ is a base for a topology $\mathscr{T}^*$ on Y and is a pseudo-base for the relative topology $\mathscr{T}(Y)$ on Y. Since each element of $\mathscr{P}(Y)$ is both open and closed in $(Y, \mathscr{T}^*)$, $(Y, \mathscr{T}^*)$ is regular and $\mathscr{P}(Y)$ is a σ-discrete base for $\mathscr{T}^*$. Therefore $(Y, \mathscr{T}^*)$ is pseudo-metrizable. Clearly, $(Y, \mathscr{T}(Y))$ is a Baire space. Therefore, since a subset of Y is $\mathscr{T}(Y)$-dense if and only if it is $\mathscr{T}^*$-dense, $(Y, \mathscr{T}^*)$ is a Baire space.

Suppose f is a real-valued function defined on X. Since 1.1 is true if the phrase "X is a metric space" is replaced by the phrase "X is a pseudo-metric space", there is a $\mathscr{T}^*$-dense subset D of Y such that $f|D$ is $\mathscr{T}^*$-continuous. Clearly D is the required set.

Next, we identify some classes of spaces which have σ-disjoint pseudo-bases.

1.8. DEFINITIONS. A topological space is called quasi-regular [9] if every nonempty open set contains the closure of a nonempty open set.

A pseudo-base $\mathscr{P}$ for X is called locally countable [9] if the set $\{P' : P' \in \mathscr{P}, P' \subset P\}$ is countable for every P in $\mathscr{P}$.

1.9. PROPOSITION. *If* $(X, \mathscr{T})$ *satisfies any of the following conditions, then* $\mathscr{T}$ *has a σ-disjoint pseudo-base.*

(1) *There is a quasi-regular space* Y *which has a dense pseudo-metrizable subspace such that* X *is a dense subset of* Y.

(2) X *has a locally countable pseudo-base* $\mathscr{P}$.

(2a) X *has a countable dense subset* D *such that there is a countable local base for* $\mathscr{T}$ *at each point of* D.

(3) X *is a semi-metrizable Baire space.*

(4) $X=\prod\{X_n : n \in N\}$, *where each* X_n *has a σ-disjoint pseudo-base.*

PROOF. (1) We may assume that X is a quasi-regular space with a dense pseudo-metrizable subspace D. Then D has a σ-disjoint base $\mathscr{B}$. For each B in $\mathscr{B}$, let $U(B)$ in $\mathscr{T}$ be such that $U(B) \cap D=B$. Then $\{U(B) : B \in \mathscr{B}\}$ is a σ-disjoint pseudo-base for $\mathscr{T}$.

(2) Let $\mathscr{U}$ denote a maximal disjoint subfamily of $\mathscr{P}$. For U in $\mathscr{U}$, let $\mathscr{P}(U)=\{P : P \in \mathscr{P}, P \subset U\}$. Then $\mathscr{P}(U)=\{P(U, n) : n \in N\}$. For n in N,

let $\mathscr{P}_n=\{P(U, n): U \in \mathscr{U}\}$. Then $\bigcup \{\mathscr{P}_n: n \in N\}$ is a σ-disjoint pseudo-base for $\mathscr{T}$.

(2a) If X is as in (2a), then X has a countable pseudo-base.

(3) Suppose d is a semi-metric compatible with $\mathscr{T}$. For x in X and $\varepsilon>0$, let $S(x, \varepsilon)$ denote the interior of the set $\{y: d(x, y)<\varepsilon\}$.

Define, by induction, a sequence $(\mathscr{P}_n)_{n\in N}$ such that, for each n in N,

(a) $\mathscr{P}_n=\{S(x, \varepsilon_n(x)): x \in F_n\}$ and $F_n \subset F_{n+1}$,

(b) $\mathscr{P}_n$ is a disjoint family, $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$, and $\bigcup \mathscr{P}_n$ is dense in X, and

(c) $\varepsilon_n(x)<2^{-n}$ and $\varepsilon_{n+1}(x)=\varepsilon_n(x)/2$ for all x in F_n.

Let $D=\bigcap \{\bigcup \mathscr{P}_n: n \in N\}$ and $F=\bigcup \{F_n: n \in N\}$. Then F is dense in D and D is dense in X. Let $\mathscr{P}=\bigcup \{\mathscr{P}_n: n \in N\}$. Since (c) holds and F is dense in X, $\mathscr{P}$ is a pseudo-base for $\mathscr{T}$.

(4) First, it follows that, for each n in N, $\prod \{X_k: k \in N, k\leqq n\}$ has a σ-disjoint pseudo-base $\mathscr{P}_n$. Suppose $\mathscr{P}_n=\bigcup \{\mathscr{P}(n,j): j \in N\}$, where each $\mathscr{P}(n,j)$ is a disjoint family. For n, j in N, let

$$\mathscr{P}^*(n,j) = \left\{P \times \prod \{X_k: k \in N, k > n\}: P \in \mathscr{P}(n,j)\right\}.$$

Then each $\mathscr{P}^*(n,j)$ is a disjoint family, and $\bigcup \{\mathscr{P}^*(n,j): n, j \in N\}$ is a pseudo-base for $\prod \{X_k: k \in N\}$.

1.10. REMARKS. (1) It follows from 1.9(1) that, if M is a metric base which is of the first category in itself, then 1.2 holds for $\beta M-M$. (Here βM denotes the Stone-Čech compactification of M.) Also, any subset of $\beta M-M$ which is metrizable is nowhere dense in $\beta M-M$.

(2) It is easy to verify that the subspace F in the proof of 1.9(3) is metrizable. Thus any semi-metrizable Baire space contains a dense metrizable subspace. This strengthens a result of H. Bennett **[2]**.

(3) It follows from 1.7 and 1.9(3) that 1.2 holds for every semi-metrizable Baire space. This result was first proven by H. Bennett **[2]**.

(4) The following statement, which is slightly more general than 1.7, is true. If $(X, \mathscr{T})$ is a Baire space, $\mathscr{P}$ is a σ-disjoint subfamily of $\mathscr{T}$ not containing $\varnothing$, and $f: X\rightarrow R$, then there is a subset D of X such that $f|D$ is continuous and $D\cap P\neq\varnothing$ for every P in $\mathscr{P}$.

1.11. PROPOSITION. *Suppose X is a quasi-regular compact space which satisfies the following condition.*

1.12. *If $(U_n)_{n\in N}$ is a sequence of open sets such that $\bigcap \{U_n: n \in N\}\neq\varnothing$, then* $\operatorname{int}[\bigcap \{U_n: n \in N\}]\neq\varnothing$. *Then* 1.4(1) *holds for* $\mathfrak{m}=\aleph_1$.

PROOF. Suppose $(G_\alpha)_{\alpha\in\Gamma}$ is a family of dense open sets such that $|\Gamma|\leqq\aleph_1$. We may assume $\Gamma=\{\alpha: \alpha<\omega_1\}$. Let U be a nonempty open set and let

H_0 be a nonempty open set such that cl $H_0 \subset U \cap G_0$. Suppose, for some γ, $1 \leqq \gamma < \omega_1$, we have defined $(H_\alpha)_{\alpha<\gamma}$ such that

(γ.1) if $\alpha < \gamma$ then H_α is a nonempty open set and cl $H_\alpha \subset U \cap G_\alpha$ and

(γ.2) if $\alpha < \alpha' < \gamma$ then cl $H_{\alpha'} \subset H_\alpha$.

Case 1. Suppose $\gamma = \delta + 1$. Then $G_\gamma \cap H_\delta$ is a nonempty open set, so there is a nonempty open set H_γ such that cl $H_\gamma \subset G_\gamma \cap H_\delta$.

Case 2. Suppose γ is a limit ordinal. Since (cl $H_\alpha)_{\alpha<\gamma}$ has the finite intersection property, $\bigcap \{\text{cl } H_\alpha : \alpha < \gamma\} \neq \varnothing$. But, since γ is a limit ordinal and (γ.2) holds, $\bigcap \{\text{cl } H_\alpha : \alpha < \gamma\} = \bigcap \{H_\alpha : \alpha < \gamma\}$. Since 1.12 holds, $\text{int}[\bigcap \{H_\alpha : \alpha < \gamma\}] \neq \varnothing$. Let H_γ be a nonempty open set such that cl $H_\gamma \subset G_\gamma \cap \text{int}[\bigcap \{H_\alpha : \alpha < \gamma\}]$.

So we have a family $(H_\alpha)_{\alpha<\omega_1}$ of nonempty open sets such that $(\omega_1 \cdot 1)$ and $(\omega_1 \cdot 2)$ hold. Since (cl $H_\alpha)_{\alpha<\omega_1}$ has the finite intersection property, $\bigcap \{\text{cl } H_\alpha : \alpha < \omega_1\} \neq \varnothing$. But

$$\bigcap \{\text{cl } H_\alpha : \alpha < \omega_1\} \subset U \cap [\bigcap \{G_\alpha : \alpha < \omega_1\}].$$

1.13. COROLLARY. *Suppose* $2^{\aleph_0} = \aleph_1$. *If X is a locally compact, Hausdorff space which satisfies* 1.12, *then* 1.2 *holds for X.*

1.14. REMARKS. (1) If D is an infinite discrete space, then $\beta D - D$ satisfies 1.12.

(2) If X is a T_1 Baire space which has no isolated points and satisfies 1.12, then X does not have a σ-disjoint pseudo-base.

2. In this section, we describe two Baire spaces for which Blumberg's theorem does not hold.

2.1. EXAMPLE. Let R denote the set of real numbers and let $\mathscr{T}$ denote the density topology on R (see [7]). It was shown in [6] that $(R, \mathscr{T})$ is a completely regular, Hausdorff space. We shall show that $(R, \mathscr{T})$ is a Baire space for which, if $2^{\aleph_0} = \aleph_1$, statement 1.2 does not hold.

We shall denote Lebesgue outer measure, Lebesgue measure, and Lebesgue inner measure by m^*, m, and m_*, respectively. Let $\mathscr{L}$ denote the family of all Lebesgue measurable subsets of R and let $\mathscr{E}$ denote the Euclidean topology on R. If $\mathscr{F}$ is a family of subsets of R and $B \subset R$, we denote by $\mathscr{F} \cap B$ the family $\{F \cap B : F \in \mathscr{F}\}$.

Suppose $A \in \mathscr{L}$ and $x \in R$. The upper density of A at x, denoted by $d^-(x, A)$, is defined to be

$$\lim(n \to \infty)\sup\left\{\frac{m(A \cap I)}{m(I)} : I \text{ a closed interval, } x \in I,\ 0 < m(I) < n^{-1}\right\}.$$

The lower density of A at x, denoted by $d\_(x, A)$, is defined similarly. If $d^-(x, A) = d\_(x, A) = \gamma$, we say that A has density γ at x and denote γ by $d(x, A)$.

Let $\mathscr{T}=\{A\in\mathscr{L}:d(x, A)=1$ for all x in $A\}$. In [6] and [7], the following statements were proven.

(1) $\mathscr{T}$ is a topology for R containing $\mathscr{E}$ such that no point in R has a countable $\mathscr{T}$-neighborhood base.

(2) A function $f: R\to R$ is $(\mathscr{T}, \mathscr{E})$-continuous if and only if it is approximately continuous [5] at every x in R.

(3) $(R, \mathscr{T})$ is a completely regular, Hausdorff space which is not normal. (There are disjoint, countable, closed sets which are not completely separated, and, even though every $\mathscr{T}$-closed subset of R is a G_δ relative to $\mathscr{T}$, there are $2^{\mathfrak{c}}$ $\mathscr{T}$-closed subsets of R and only $\mathfrak{c}$ subsets of R which are zero sets relative to $\mathscr{T}$.)

(4) If $A\in\mathscr{T}$, then $(A, \mathscr{T}\cap A)$ is connected if and only if $A\in\mathscr{E}$ and $(A, \mathscr{E}\cap A)$ is connected.

If $A\in\mathscr{L}$, then it follows from the Lebesgue density theorem [5, p. 174] that $\mathscr{T}$-int $A=\{x\in A:d(x, A)=1\}$. So, a subset D of R is $\mathscr{T}$-dense in R if and only if $m_*(R-D)=0$. Therefore the intersection of a countable family of $\mathscr{T}$-dense elements of $\mathscr{T}$ is a $\mathscr{T}$-dense element of $\mathscr{T}$.

2.1.1. *Suppose $f: R\to R$ and A is a subset of R such that $m^*(A)>0$ and $f|A$ is $(\mathscr{T}\cap A, \mathscr{E})$-continuous. Then there is an uncountable set K contained in A such that $f|K$ is $(\mathscr{E}\cap K, \mathscr{E})$-continuous.*

PROOF. Since $f|A$ is $(\mathscr{T}\cap A, \mathscr{E})$-continuous and $\mathscr{T}\subset\mathscr{L}$, $f|A$ is measurable $(\mathscr{L}\cap A)$. Let μ^* denote the restriction of m^* to the family of all subsets of A. Then $\mathscr{L}\cap A$ is the family of all μ^*-measurable subsets of A. Let $\mu=\mu^*|(\mathscr{L}\cap A)$. If $S\in\mathscr{L}\cap A$ and $\varepsilon>0$, then there is an $(\mathscr{E}\cap A)$-closed subset F of A such that $F\subset S$ and $\mu(S-F)<\varepsilon$. Hence Lusin's theorem holds for $(A, \mathscr{L}\cap A, \mathscr{E}\cap A, \mu)$. Therefore, since $\mu(A)=m^*(A)>0$, there is K in $\mathscr{L}\cap A$ such that $m^*(K)=\mu(K)>0$ and $f|K$ is $(\mathscr{E}\cap K, \mathscr{E})$-continuous. Since $m^*(K)>0$, K is uncountable.

2.1.2. *If $2^{\aleph_0}=\aleph_1$, then there is a function $f: R\to R$ such that, if $A\subset R$ and $f|A$ is $(\mathscr{T}\cap A, \mathscr{E})$-continuous, then $m(A)=0$.*

PROOF. If $2^{\aleph_0}=\aleph_1$, then there is a function $f: R\to R$ such that, if $A\subset R$ and $f|A$ is $(\mathscr{E}\cap A, \mathscr{E})$-continuous, then A is countable [5, p. 148]. By 2.1.1, if $f|A$ is $(\mathscr{T}\cap A, \mathscr{E})$-continuous, $m^*(A)=0$.

It follows from 2.1.2 that, if $2^{\aleph_0}=\aleph_1$, then Blumberg's theorem does not hold for $(R, \mathscr{T})$.

It might be conjectured that 1.2 holds for certain subclasses of the class of Baire spaces. Dr. B. J. Pettis, in a letter, suggested that perhaps 1.2 holds for all cocompact [1] spaces, or for all α-favorable [4, p. 116] spaces, or for all paracompact Baire spaces. We shall show that, if $2^{\aleph_0}=\aleph_1$, none of these conjectures is true. We shall show that $(R, \mathscr{T})$ is cocompact, strongly α-favorable [4, p. 117], and pseudo-complete [8, p. 164].

And, in 2.2, we shall give an example of a hereditarily Lindelöf, Baire space for which Blumberg's theorem does not hold.

We need the following theorem which was proven in [**6**].

2.1.3. Lusin-Menchoff Theorem. *Suppose $A \in \mathscr{L}$ and F is an $\mathscr{E}$-closed set such that $F \subset \mathscr{T}$-int A. Then there is an $\mathscr{E}$-closed set P such that $F \subset \mathscr{T}$-int $P \subset P \subset A$.*

2.1.4. *$(R, \mathscr{T})$ is cocompact.*

Proof. Let $\mathscr{T}^* = \{U \in \mathscr{E} : (R-U, \mathscr{E} \cap (R-U))$ is compact$\} \cup \{\varnothing\}$. It is clear that $(R, \mathscr{T}^*)$ is compact and $\mathscr{T}^* \subset \mathscr{T}$. And $\mathscr{T}^*$ is a cotopology for $(R, \mathscr{T})$. For suppose $x \in A \in \mathscr{T}$ and A is a bounded subset of R. By 2.1.3, there is an $\mathscr{E}$-closed set P such that $x \in \mathscr{T}$-int $P \subset P \subset A$. Since $(P, \mathscr{E} \cap P)$ is compact, P is $\mathscr{T}^*$-closed.

It follows that $(R, \mathscr{T})$ is strongly α-favorable since any regular, cocompact space is strongly α-favorable.

2.1.5. *$(R, \mathscr{T})$ is pseudo-complete.*

Proof. For each n in N, let $\mathscr{B}_n$ denote the family of all nonempty elements U of $\mathscr{T}$ such that, if $U^* = \mathscr{E}$-cl U, $a(U) = \inf U^*$, and $b(U) = \sup U^*$, then (1) $(U^*, \mathscr{E} \cap U^*)$ is compact, (2) $b(U) - a(U) < 1/n$, and

$$m(U \cap [a(U), b(U)])/(b(U) - a(U)) > 1 - n^{-1}. \tag{3}$$

It is easily verified that each $\mathscr{B}_n$ is a base for $\mathscr{T}$.

Now suppose that $(U_n)_{n \in N}$ is a sequence such that, for each n in N, $U_n \in \mathscr{B}_n$ and $\mathscr{T}$-cl $U_{n+1} \subset U_n$. By (1), $\bigcap \{\mathscr{E}\text{-cl } U_n : n \in N\} \neq \varnothing$. By (2), $\lim(n \to \infty)(b(U_n) - a(U_n)) = 0$. Hence $\bigcap \{\mathscr{E}\text{-cl } U_n : n \in N\}$ contains only one point, say x_0. Suppose $n \in N$. By (3),

$$\lim(k \to \infty) \frac{m(U_n \cap [a(U_k), b(U_k)])}{b(U_k) - a(U_k)} = 1.$$

Therefore $d^{-}(x_0, U_n) = 1 > 0$. Hence $x_0 \in \mathscr{T}$-cl U_n and

$$x_0 \in \bigcap \{\mathscr{T} - \text{cl } U_n : n \in N\} = \bigcap \{U_n : n \in N\}.$$

2.2. Example. Suppose $2^{\aleph_0} = \aleph_1$. Let $\mathscr{M}$ be a disjoint subfamily of $\mathscr{L}$ maximal with respect to the property: if $A \in \mathscr{M}$, then there is a subset $S(A)$ of A such that (1) $m^*(S(A)) > 0$, (2) A is a measurable cover of $S(A)$, and (3) if $Z \subset S(A)$ and $m(Z) = 0$, then $|Z| \leqq \aleph_0$. Let $S = \bigcup \{S(A) : A \in \mathscr{M}\}$. Since $|\mathscr{M}| \leqq \aleph_0$, (3), with $S(A)$ replaced by S, holds. Since every A in $\mathscr{L}$ of positive measure contains a subset $S(A)$ of cardinality $\aleph_1$ for which (3) holds [5, p. 168], $m_*(R-S) = 0$. Hence S is $\mathscr{T}$-dense in R.

$(S, \mathscr{T} \cap S)$ is a Baire space; in fact, the intersection of a countable family of open, dense subsets of $(S, \mathscr{T} \cap S)$ is an open, dense subset of $(S, \mathscr{T} \cap S)$. By 2.1.2, Blumberg's theorem does not hold for $(S, \mathscr{T} \cap S)$. Finally, it follows from the next statement that $(S, \mathscr{T} \cap S)$ is hereditarily Lindelöf.

2.2.1. *If $\mathscr{U} \subset \mathscr{T}$, then there is a countable subfamily $\mathscr{C}$ of $\mathscr{U}$ such that $m(\bigcup \mathscr{U} - \bigcup \mathscr{C}) = 0$.*

PROOF. Let $U = \bigcup \mathscr{U}$. We may assume that U is a bounded set; say $U \subset (-k, k)$, where $k \in N$. Let $\varepsilon > 0$. The family $\mathscr{V}$, of all closed intervals I contained in $(-k, k)$ for which there is $U(I)$ in U such that

$$m(I \cap U(I))/m(I) > 1 - 2\varepsilon k^{-1},$$

covers U in the sense of Vitali [**5**, p. 170]. So, by Vitali's theorem [**5**, p. 170], there is a disjoint subfamily $\mathscr{F}$ of $\mathscr{V}$ such that $m(U - \bigcup \mathscr{F}) = 0$. Let $\mathscr{C}(\varepsilon) = \{U(I) : I \in \mathscr{F}\}$. Then $m^*(U - \bigcup \mathscr{C}(\varepsilon)) < \varepsilon$. Let

$$\mathscr{C} = \bigcup \{\mathscr{C}(n^{-1}) : n \in N\}.$$

2.3. REMARKS. (1) It follows from 2.2.1 that $(R, \mathscr{T})$ is weakly Lindelöf.

(2) Any subset A of R such that $(A, \mathscr{T} \cap A)$ is Čech complete is $\mathscr{T}$-nowhere dense and $(A, \mathscr{T} \cap A)$ is discrete. For, if $m^*(A) > 0$, then A has at least one nonisolated point, and every countable subset of A is $\mathscr{T}$-closed.

(3) Any subset A of S such that $(A, \mathscr{T} \cap A)$ is Čech complete is both countable and $\mathscr{T} \cap A$-nowhere dense.

(4) Even though $|\mathscr{T}| = 2^{\mathfrak{c}}$, the cardinality of $\mathscr{T} \cap S$ is $\mathfrak{c}$.

(5) Every $\mathscr{T}$-open subset of R is an F_σ relative to $\mathscr{T}$; hence $(S, \mathscr{T} \cap S)$ is hereditarily perfectly normal.

(6) It would be interesting to know whether 1.2 holds for every compact, Hausdorff space.

2.4. The author has been informed that R. F. Levy [**8**] has discovered, independently, an example of a completely regular, Hausdorff Baire space for which Blumberg's theorem does not hold.

3. If, in Theorem 1 of J. B. Brown's paper entitled *Metric spaces in which a strengthened form of Blumberg's theorem holds*, Fund. Math. **71** (1971), 243–253, the phrase "X is a metric space" is replaced by the phrase "X is a space with a σ-disjoint pseudo-base", then the resulting statement is true. The proof of this is quite similar to the proof of 1.7.

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