ON A SUBCLASS OF SPIRAL-LIKE FUNCTIONS

E. M. SILVIA

Abstract. Let \( \alpha \geq 0 \), \( 0 \leq \beta < 1 \), \( |\lambda| < \pi/2 \) and suppose that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is holomorphic in \( U = \{ z : |z| < 1 \} \). If

\[
\text{Re} \left[ e^{i\lambda} \frac{zf''(z)}{f'(z)} + \alpha \left( \frac{zf''(z)}{f'(z)} + 1 - \frac{zf''(z)}{f(z)} \right) \right] > \beta \cos \lambda
\]

for \( z \in U \), then \( f(z) \) is said to be \( \alpha-\lambda \)-spiral-like of order \( \beta \) and we write \( f(z) \in S_\alpha^\lambda(\beta) \). The author shows that for each \( \alpha \geq 0 \), \( \alpha-\lambda \)-spiral-like functions of order \( \beta \) are \( \lambda \)-spiral-like of order \( \beta \). The following representation theorem is obtained: The function \( f(z) \in S_\alpha^\lambda(\beta) \) \( (\alpha > 0, 0 \leq \beta < 1, |\lambda| < \pi/2) \), if and only if there exists a function \( F(\zeta) \) \( \lambda \)-spiral-like of order \( \beta \) such that

\[
F(z) = \left[ (e^{i\lambda}/\alpha) \int_0^{\alpha} F(\zeta) e^{i\lambda/\alpha} \frac{d\zeta}{\zeta} \right] z e^{-i\lambda}.
\]

A distortion theorem for \( \log |f(z)/z| \) and a rotation theorem for \( \arg f(z)/z \) are also proved for functions \( f(z) \in S_\alpha^\lambda(\beta) \).

1. Let \( A \) denote the class of functions normalized by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) which are analytic in \( U \) \( (|z| < 1) \). For \( 0 \leq \beta < 1 \), we will let \( S^*(\beta) \) represent the class of functions contained in \( A \) which are univalent and starlike of order \( \beta \); i.e., \( f(z) \in S^*(\beta) \) if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic and univalent satisfying \( \text{Re} zf''(z)/f(z) > \beta \ (z \in U) \). Also, let \( P \) denote the class of analytic functions normalized by \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) such that \( \text{Re} p(z) > 0 \ (z \in U) \).

A function \( f(z) \in A \) is said to be spiral-like if there exists a \( \lambda \) \( (|\lambda| < \pi/2) \) such that \( \text{Re} e^{i\lambda}zf''(z)/f(z) > 0 \ (z \in U) \). L. Spaček defined the class of spiral-like functions in 1933 and showed that these functions are univalent [15].

In 1967, R. Libera [6] extended this definition to functions spiral-like of order \( \beta \). We say that \( f(z) \in A \) is \( \lambda \)-spiral-like of order \( \beta \) \( (0 \leq \beta < 1, |\lambda| < \pi/2) \) if \( \text{Re} e^{i\lambda}zf''(z)/f(z) > \beta \cos \lambda \ (z \in U) \).

A function \( f(z) \in A \) satisfying \( f(z)f''(z) \neq 0 \) \( (0 < |z| < 1) \) is said to be \( \alpha \)-starlike of order \( \beta \) \( (\alpha \geq 0, 0 \leq \beta < 1) \) if

\[
\text{Re} \left( 1 - \alpha \right) \frac{zf''(z)}{f(z)} + \alpha \left( \frac{zf''(z)}{f'(z)} + 1 \right) > \beta \quad (z \in U).
\]
For \( \beta = 0 \), we have the class of \( \alpha \)-starlike functions (of order zero) which has been thoroughly investigated in [7], [8], [9], [10], and [11]. Some of these results have been extended to \( 0 < \beta < 1 \) by the author [13].

In this note, a class of functions which contains the classes of \( \alpha \)-starlike functions of order \( \beta \) and \( \lambda \)-spiral-like functions of order \( \beta \) as special cases is defined; the functions in this new class will be shown to be \( \lambda \)-spiral-like. The author obtains an integral representation for the elements of this class in terms of \( \lambda \)-spiral-like functions of order \( \beta \). Finally, a distortion and a rotation theorem for \( f(z)/z \) whenever \( f(z) \) is in this class is proved.

2. Just as the definition of \( \lambda \)-spiral-likeness of order \( \beta \) generalizes the definition of starlikeness of order \( \beta \), we will generalize the definition of \( \alpha \)-starlikeness of order \( \beta \) to \( \alpha \)-\( \lambda \)-spiral-likeness of order \( \beta \). In this section, we define the class of \( \alpha \)-\( \lambda \)-spiral-like functions of order \( \beta \)—denoted \( S(\beta) \)—and show that each \( f(z) \in S(\beta) \) is \( \lambda \)-spiral-like of order \( \beta \).

**Definition 1.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A \) and satisfy \( f(z) f'(z) \neq 0 \) in \( 0 < |z| < 1 \). Set

\[
K(\lambda, \alpha, f(z)) = (e^{i\lambda} - \alpha) z f'(z) / f(z) + \alpha (z f''(z) / f'(z) + 1).
\]

Then \( f(z) \) is said to be \( \alpha \)-\( \lambda \)-spiral-like of order \( \beta \) if

\[
\text{Re} K(\lambda, \alpha, f(z)) > \beta \cos \lambda \quad (z \in U)
\]

where \( \alpha \geq 0, 0 \leq \beta < 1, |\lambda| < \pi/2 \).

**Remarks.**
(i) For \( \alpha = 0 \), \( S_0(\beta) \) is the class of \( \lambda \)-spiral-like functions of order \( \beta \).
(ii) For \( \lambda = 0 = \beta \), we have \( S_0(0) \)—the class of \( \alpha \)-star-like functions (of order zero); while \( S_0(\beta) \) (\( \alpha \geq 0, 0 \leq \beta < 1 \)) is the class of \( \alpha \)-starlike functions of order \( \beta \).

In order to prove that \( \alpha \)-\( \lambda \)-spiral-likeness of order \( \beta \) (\( \alpha \geq 0 \)) implies \( \lambda \)-spiral-likeness of order \( \beta \), we will need the following two lemmas: the first lemma is due to I. S. Jack [4] while the second is due to R. Libera [6].

**Lemma A.** Let \( \omega(z) \) be regular in \( U \) with \( \omega(0) = 0 \). If there exists a \( \zeta \in U \) such that \( \text{Max}_{|z| \leq k} |\omega(z)| = |\omega(\zeta)| \), then \( \zeta \omega'(\zeta) = k \omega(\zeta) \) for some \( k \geq 1 \).

**Lemma B.** The function \( f(z) \in A \) is \( \lambda \)-spiral-like of order \( \beta \) (\( 0 \leq \beta < 1, |\lambda| < \pi/2 \)) if and only if there exists an \( \omega(z) \) analytic satisfying \( \omega(0) = 0, |\omega(z)| < 1 \) such that

\[
e^{i\lambda} z f'(z) / f(z) = \beta \cos \lambda + (1 - \beta) \cos \lambda \left( \frac{1 - \omega(z)}{1 + \omega(z)} \right) + i \sin \lambda \quad (z \in U).
\]
THEOREM 1. If \( f(z) \in S^*_x(\beta) \) \( (\alpha \geq 0, 0 \leq \beta < 1, |\lambda| < \pi/2) \) then \( f(z) \) is \( \lambda \)-spiral-like of order \( \beta \).

PROOF. Let

\[
e^{i\lambda} \frac{zf'(z)}{f(z)} = \beta \cos \lambda + (1 - \beta) \cos \lambda \left( \frac{1 - \omega(z)}{1 + \omega(z)} \right) + i \sin \lambda.
\]

Clearly, \( \omega(0) = 0 \). In view of Lemma B, it suffices to show that \( |\omega(z)| < 1 \).

Simplifying (3), it follows that

\[
e^{i\lambda} \frac{zf'(z)}{f(z)} = \frac{e^{i\lambda} \left( 1 + (2\beta e^{-i\lambda} \cos \lambda - e^{-2i\lambda}) \omega(z) \right)}{1 + \omega(z)}.
\]

Differentiating (4) and using (1), we have

\[
K(\lambda, \alpha, f(z)) = \beta \cos \lambda + (1 - \beta) \cos \lambda \left( \frac{1 - \omega(z)}{1 + \omega(z)} \right) + i \sin \lambda
\]

(5)

\[
+ \alpha \frac{\{2\beta e^{-i\lambda} \cos \lambda - e^{-2i\lambda}\} \omega'(z)}{1 + (2\beta e^{-i\lambda} \cos \lambda - e^{-2i\lambda}) \omega(z)} - \alpha \frac{\omega'(z)}{1 + \omega(z)}.
\]

Suppose that there exists a \( \zeta \in U \) such that \( \max_{|z| \leq |\zeta|} |\omega(z)| = |\omega(\zeta)| = 1 \).

Clearly \( \omega(\zeta) \neq -1 \). From Lemma A, there exists a \( k \geq 1 \) such that

\[
\zeta \omega'(\zeta) = k \omega(\zeta).
\]

For this \( \zeta \), we have

\( K(\lambda, \alpha, f(z)) = \beta \cos \lambda \), \( |\omega(\zeta)| = k/2 \).

Also, for

\[
m = 2\beta e^{-i\lambda} \cos \lambda - e^{-2i\lambda},
\]

(7)

\[
\text{Re} \frac{m \zeta \omega'(\zeta)}{1 + m \omega(\zeta)} = \text{Re} \frac{k(|m|^2 + m \omega(\zeta))}{1 + |m|^2 + 2 \text{Re} m \omega(\zeta)}
\]

\[
= \text{Re} \frac{k(|m|^2 + \text{Re} m \omega(\zeta))}{1 + |m|^2 + 2 \text{Re} m \omega(\zeta)}.
\]

Hence,

\[
\text{Re} \left( \frac{m \zeta \omega'(\zeta)}{1 + m \omega(\zeta)} \right) - \text{Re} \left( \frac{\zeta \omega'(\zeta)}{1 + \omega(\zeta)} \right) = \frac{k(|m|^2 - 1)}{2(1 + 2 \text{Re} m \omega(\zeta) + |m|^2)}.
\]

Thus, from (6), (7) and (8), it follows that

\[
\text{Re} K(\lambda, \alpha, f(z)) = \beta \cos \lambda - \frac{2k\beta(1 - \beta) \alpha \cos^2 \lambda}{1 + |m|^2 + 2 \text{Re} m \omega(\zeta)} < \beta \cos \lambda,
\]

\[\]
contradicting the assumption that \( f(z) \in S^\lambda_\alpha(\beta) \). Therefore \(|\omega(z)| < 1\) in \( U \) and \( f(z) \) is \( \lambda \)-spiral-like of order \( \beta \).

**Corollary.** If \( f(z) \in S^\lambda_\alpha(\beta) \) then \( f(z) \in S^\gamma_\beta(0, \alpha) \), \( 0 \leq \gamma \leq \alpha \).

**Proof.** By Theorem 1, \( f(z) \in S^\lambda_\alpha(\beta) \). Suppose there exists a \( \gamma \), \( 0 < \gamma < \alpha \), such that \( f(z) \notin S^\gamma_\beta(0, \alpha) \). Then there is a \( \zeta \in U \) for which

\[
\text{Re} \left( \frac{zf''(\zeta)}{f'(\zeta)} + 1 - \frac{\zeta f'(\zeta)}{f(\zeta)} \right) \leq \frac{\beta \cos \lambda}{\gamma} - \frac{1}{\gamma} \text{Re} \frac{\zeta f'(\zeta)}{f(\zeta)}.
\]

However, for \( f(z) \in S^\lambda_\alpha(\beta) \),

\[
0 < -\beta \cos \lambda + \text{Re} e^{i\lambda} \frac{\zeta f'(\zeta)}{f'(\zeta)} + \alpha \text{Re} \left( \frac{\zeta f''(\zeta)}{f'(\zeta)} + 1 - \frac{\zeta f'(\zeta)}{f(\zeta)} \right).
\]

Substituting (10) into (11), we obtain

\[
0 < (1 - \alpha/\gamma)(\text{Re} e^{i\lambda} \zeta f'(\zeta)/f(\zeta) - \beta \cos \lambda).
\]

But \( (1 - \alpha/\gamma) < 0 \) implies \( \text{Re} e^{i\lambda} f'(z)/f(z) < \beta \cos \lambda \), contradicting the assumption that \( f(z) \in S^\lambda_\alpha(\beta) \). Thus, \( f(z) \in S^\lambda_\beta(0, \alpha) \).

3. In this section, the author obtains an important integral representation for the elements of \( S^\alpha_\lambda(\beta) \). Throughout this section \( \alpha, \beta, \lambda \) will represent constants such that \( \alpha > 0, 0 \leq \beta < 1, |\lambda| < \pi/2 \).

**Definition 2.** The function

\[
\frac{1}{(y + ip \zeta + it)} f(z) = \left[ (y + i\mu) \int_0^z \sigma(t) t^{-1-i\mu} dt \right]^{1/(y+i\mu)}
\]

where \( \sigma(t) \in S^*(0) \), \( \gamma > 0 \), \( \mu \) real, \( z \in U \) and the powers are meant as principal values, is called a Bazilevič function of type \( y + ip \). Denote the class of such functions by \( B(y + ip) \) [2].

Due to a result by Eenigenburg et al. [3], we know that each \( f(z) \in B(y + ip) \) is spiral-like. The functions that we will use in order to characterize the elements of \( S^\alpha_\lambda(\beta) \) are those obtained when \( \gamma = (\cos \lambda)/\alpha \) and \( \mu = (\sin \lambda)/\alpha \).

**Definition 3.** A function \( f(z) \in A \) is said to be a Bazilevič function of type \( e^{i\lambda}/\alpha \) and order \( \beta \) if

\[
f(z) = \left[ \frac{e^{i\lambda}}{\alpha} \int_0^z \sigma(\zeta)^{\cos \lambda}/\zeta^{-1+((i\sin \lambda)/\alpha)} d\zeta \right]^{1/(e^{i\lambda})}.
\]

for some \( \sigma(\zeta) \in S^*(\beta) \). Denote this by \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \).

Immediate from Definition 3 is

**Theorem 2.** If \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \) then \( f(z) \in S^\lambda_\alpha(\beta) \).
ON A SUBCLASS OF SPIRAL-LIKE FUNCTIONS

PROOF. For \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \), it follows from (12) that

\[
f'(z) = \sigma(z)e^{\lambda/\alpha}z^{\beta - 1 + ((\sin \lambda)/\alpha)f(z)}f(z)^{1-e^{i\lambda}/\alpha}.
\]

Taking the logarithmic derivative of (13) we obtain an expression for \([zf''(z)/f'(z)] + 1\). Substituting this into (1), we have

\[
K(\lambda, \alpha, f(z)) = \cos \lambda \sigma'(z)/\sigma(z) + i \sin \lambda.
\]

Thus, \( \Re K(\lambda, \alpha, f(z)) > \beta \cos \lambda \) or \( f(z) \in S^\lambda_\alpha(\beta) \).

Using the following lemma due to Basgöze and Keogh [1], a necessary and sufficient condition for \( f(z) \) to be in \( B(e^{i\lambda}/\alpha, \beta) \) is obtained.

**Lemma C.** A function \( \sigma(z) \in S^*(\beta) \) if and only if there exists a function \( F(\zeta) \in S^\lambda_\alpha(\beta) \) such that

\[
(\sigma(\zeta)/\zeta)^{\cos \lambda} = (F(\zeta)/\zeta)^{e^{i\lambda}} \quad (\zeta \in U).
\]

**Lemma 1.** A function \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \) if and only if there exists a function \( F(\zeta) \in S^\lambda_\alpha(\beta) \) such that

\[
f(z) = \left[ e^{i\lambda} \int_a^z \left[ \frac{F(\zeta)}{\zeta} \right]^{(\cos \lambda)/\alpha} \zeta^{\beta - 1 + (\sin \lambda)/\alpha} d\zeta \right]^{ae^{-i\lambda}}
\]

where the powers are meant as principal values.

**Proof.** From Definition 3, \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \) if and only if there exists a \( \sigma(\zeta) \in S^*(\beta) \) satisfying (12). However, a necessary and sufficient condition for \( \sigma(\zeta) \in S^*(\beta) \) is that there exists an \( F(\zeta) \in S^\lambda_\alpha(\beta) \) satisfying (15). Thus, for \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \), we may obtain

\[
f(z) = \left[ e^{i\lambda} \int_a^z \left( \frac{\sigma(\zeta)}{\zeta} \right)^{(\cos \lambda)/\alpha} \zeta^{\beta - 1 + (\sin \lambda)/\alpha} d\zeta \right]^{ae^{-i\lambda}}
\]

where \( \sigma(\zeta) \) and \( F(\zeta) \) are as above. Since each step in (17) is reversible, the result follows from this identity.

**Remark.** From Lemma 1, a necessary and sufficient condition for \( f(z) \in B(e^{i\lambda}/\alpha, \beta) \) is that

\[
F(z) = f(z)[zf''(z)/f'(z)]^{ae^{-i\lambda}}
\]

where \( F(z) \in S^\lambda_\alpha(\beta) \). Also, \( B(e^{i\lambda}/\alpha, \beta) \subseteq S^\lambda_\alpha(\beta) \). In order to obtain the
characterization for functions \( f(z) \in S_0^A(\beta) \), we consider the converse problem. Given \( F(\zeta) \in S_0^A(\beta) \) and \( \alpha > 0 \), when is the solution to the differential equation (18) with boundary condition \( f(0)=0 \), a function that is \( \alpha-\lambda \)-spiral-like of order \( \beta \)? Since (18) may be rewritten as \( [F(z)]^{e^{-i\lambda} / \alpha} / z = -f'(z)f(z)^{-1+(e^{i\lambda} / \alpha)} \) we may perform the integration with boundary condition \( f(0)=0 \) to obtain
\[
f(z) = \left[ \frac{e^{i\lambda}}{\alpha} \int_0^z \frac{[F(\zeta)]^{e^{i\lambda} / \alpha}}{\zeta} d\zeta \right] z^{e^{-i\lambda}}.
\]

We will now show the proper definitions for which this formal solution is indeed an \( \alpha-\lambda \)-spiral-like function of order \( \beta \).

**Lemma 2.** Let \( f(z) \in S_0^A(\beta) \). For \( 0 < \gamma \leq z \), choose the branch of \( [zf'(z)f(z)]^{e^{-i\lambda}} \) equal to 1 when \( z=0 \). Then the function
\[
F_\gamma(z) = f(z)[zf'(z)/f(z)]^{e^{-i\lambda}}
\]
is \( \lambda \)-spiral-like of order \( \beta \).

**Proof.** We have
\[
e^{i\lambda} zF'(z) = e^{i\lambda} zF'(z) + \gamma \left( \frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)} \right) = K(\lambda, \gamma, f(z)).
\]
But by the corollary to Theorem 1, we have that \( f(z) \in S_0^A(\beta) \) implies 
\( f(z) \in S_0^A(\beta) \) \( 0 \leq \gamma \leq z \). Therefore, \( Re \ e^{i\lambda} zF'(z)/F_\gamma(z) = Re K(\lambda, \gamma, f(z)) \geq \beta \cos \lambda \) and \( F_\gamma(z) \in S_0^A(\beta) \).

**Lemma 3.** If \( F(z)=z+A_2z+\cdots \in S_0^A(\beta) \) then \( F(z) \) may be expressed as
\[
F(z) = f(z)[zf'(z)/f(z)]^{e^{-i\lambda}},
\]
where
\[
f(z) = \left[ \frac{e^{i\lambda}}{\alpha} \int_0^z [F(\zeta)]^{e^{i\lambda} / \alpha} \zeta d\zeta \right] z^{e^{-i\lambda}}
\]
is an \( \alpha-\lambda \)-spiral-like function of order \( \beta \).

**Proof.** Let \( h(z)=z^{-e^{i\lambda} / \alpha} \int_0^z [F(\zeta)]^{e^{i\lambda} / \alpha} \zeta d\zeta \). We have
\[
f(z) = z \left[ (e^{i\lambda} / \alpha) h(z) \right]^{e^{-i\lambda}}
\]
so that if \( h(z) \) is independent of the path of integration it will follow that \( f(z) \) is well defined.
Since \( F(z)=z(1+A_2z+\cdots) \in S_0^A(\beta) \), we have that \( (1+A_2z+\cdots) \) is
nonzero in $U$. Thus, we may write

$$\sum_{n=1}^{\infty} c_n z^n$$

for the power series expansion about $z=0$. From (22), it follows that

$$\int_0^z F(\zeta) e^{i\lambda/\zeta-1} d\zeta = \alpha e^{-i\lambda} e^{i\lambda/\sqrt{1 + \frac{\sum_{n=1}^{\infty} c_n}{\alpha e^{i\lambda} + 1}}} z^n + C.$$  

To obtain a solution of (23) which is analytic and zero at the origin, take $C=0$. Thus, $h(z) = \alpha e^{-i\lambda} (1 + \sum_{n=1}^{\infty} c_n z^n / (\alpha e^{i\lambda} + 1))$ is independent of the path of integration so that $f(z)$ given by (21) is well defined.

That $f(z)$ is $\alpha$-$\lambda$-spiral-like of order $\beta$ is a consequence of Theorem 2 and Lemma 1. Thus, the lemma is proved.

By combining the results of Theorem 2, Lemma 2, and Lemma 3, we have

**Theorem 3.** A necessary and sufficient condition for $f(z)$ to be in $S^\lambda_\beta(\alpha)$ is that $f(z)$ have the integral representation

$$f(z) = \left[ e^{i\lambda} \int_0^z [F(\zeta)] e^{i\lambda/\sqrt{1 + \frac{\sum_{n=1}^{\infty} c_n}{\alpha e^{i\lambda} + 1}}} d\zeta \right] e^{-i\lambda}$$

for some $F(\zeta) \in S^\lambda_\beta(\alpha)$, where the powers are assumed to be principal values.

**Proof.** If $f(z)$ is of the form (24), it follows immediately from Theorem 2 and Lemma 1 that $f(z) \in S^\lambda_\beta(\alpha)$. If $f(z) \in S^\lambda_\beta(\alpha)$, then—by Lemma 2 and Lemma 3—$f(z)$ can be written in the form (24).

Note that we now have $S((\alpha, \beta) = S^\lambda_\beta(\alpha)$ for $\alpha > 0$, $0 \leq \beta < 1$, $|\lambda| < \pi/2$.

4. We conclude this paper with a determination of a distortion theorem and a rotation theorem for $f(z)/z$ whenever $f(z) \in M^\lambda_\beta(\alpha) = M^\lambda_\beta(\alpha)$ (0 $\leq \beta < 1$, $|\lambda| < \pi/2$).

For $f(z) \in M^\lambda_\beta(\alpha)$ (0 $\leq \beta < 1$, $|\lambda| < \pi/2$) there exists a $p(z) \in P$ such that

$$e^{i\lambda z} f'(z)/f(z) = (1 - \beta) \cos \lambda p(z) + \beta \cos \lambda + \beta \sin \lambda.$$

From (25) it follows that

$$e^{i\lambda z} f'(z)/f(z) - 1 = (1 - \beta) \cos \lambda (p(z) - 1).$$

Throughout this section $\lambda$, $\beta$ will denote constants satisfying $|\lambda| < \pi/2$, $0 \leq \beta < 1$.

Using (26) we are able to obtain the convex hull of the image of $\log f(z)/z$ for fixed $z$ ($|z| = r < 1$) when $f(z) \in M^\lambda_\beta(\alpha)$.

**Theorem 4.** If $f(z) \in M^\lambda_\beta(\alpha)$ then the set of all possible values of $\log f(z)/z$ (z fixed, $|z| = r < 1$) lies in the image of $|z| \leq r$ under the map

$$\omega(z) = \log[(1 - \varepsilon z)^{-2(1 - \beta) + \cos \lambda}], \quad |\varepsilon| = 1.$$
PROOF. Dividing both sides of (26) by \( z \neq 0 \), integrating from 0 to \( z \) and simplifying, we have

\[
\log \frac{f(z)}{z} = (1 - \beta)e^{-i\lambda} \cos \lambda \int_0^z \frac{p(\zeta) - 1}{\zeta} \, d\zeta.
\]

Since \( p(z) \in P \), Herglotz's theorem [12] may be applied to obtain

\[
(27) \quad \frac{p(\zeta) - 1}{\zeta} = (1 - \beta)e^{-i\lambda} \cos \lambda \int_{-\pi}^\pi \frac{2e^{it}}{1 - e^{it}} \, d\mu(t),
\]

where \( \mu(t) \) is nondecreasing in \([\pi, \pi]\) and \( \int_{-\pi}^\pi d\mu(t) = 1 \). From (28), it follows that

\[
\frac{p(\zeta) - 1}{\zeta} = \int_{-\pi}^\pi \frac{2e^{it}}{1 - e^{it}} \, d\mu(t).
\]

Substituting (29) into (27), we obtain

\[
\log \frac{f(z)}{z} = -2(1 - \beta)e^{-i\lambda} \cos \lambda \int_{-\pi}^\pi \log(1 - e^{it}z) \, d\mu(t).
\]

Let \( q(z, t) = \log(1 - e^{it}z)^{-2(1 - \beta)}e^{-i\lambda} \cos \lambda \). Then \( \text{Re} \{1 + zq''(z, t)/q'(z, t)\} = \text{Re} [1/(1 - ze^{it})] > \frac{1}{2} \). Thus, \( q(z, t) \) is univalent in \( z \) and maps \( |z| \leq r < 1 \) onto a convex domain \( E \), independent of \( t \). From (30), we know that for fixed \( z \) \((|z|=r<1)\) the points of \( \log f(z)/z \) lie in the convex hull of \( E \), denoted \( \text{con} \ E \). However, since \( E \) is convex, \( E = \text{con} \ E \) and the points of \( \log f(z)/z \) \((z \text{ fixed}, |z|=r<1)\) lie in the convex image of \( |z| \leq r \) under the mapping \( \omega(z) \) given by (27).

REMARKS. (i) For

\[
\log f_t(z)/z = \log[(1 - e^{it}z)^{-2(1 - \beta)}e^{-i\lambda} \cos \lambda] \quad (-\pi \leq t < \pi),
\]

we have

\[
f_t(z) = z(1 - e^{it}z)^{-2(1 - \beta)}e^{-i\lambda} \cos \lambda.
\]

These \( f_t(z) \)---for different \( t \)---are the extremal functions for Theorem 4.

(ii) We have

\[
(31) \quad \log |f_t(z)/z| = \text{Re} \log[(1 - e^{it}z)^{-2(1 - \beta)}e^{-i\lambda} \cos \lambda]
\]

and

\[
(32) \quad \arg f_t(z)/z = \text{Im} \log[(1 - e^{it}z)^{-2(1 - \beta)}e^{-i\lambda} \cos \lambda].
\]

Also, for \( z = re^{i\theta} \) \((0 < r < 1, 0 \leq \theta < 2\pi)\) and \( \eta = \theta + t \), we have

\[
(33) \quad \log[(1 - e^{it}z)^{-2(1 - \beta)}e^{-i\lambda} \cos \lambda] = T(r, \eta, \lambda, \beta) + iS(r, \eta, \lambda, \beta)
\]
where

\[ T(r, \eta, \lambda, \beta) \]

\[
= (1 - \beta) \cos \lambda \left\{ 2 \sin \lambda \arctan \frac{r \sin \eta}{1 - r \cos \eta} - \cos \lambda \log(1 - 2r \cos \theta + r^2) \right\}
\]

and

\[ S(r, \eta, \lambda, \beta) \]

\[
= (1 - \beta) \cos \lambda \left\{ 2 \cos \lambda \arctan \frac{r \sin \eta}{1 - r \cos \eta} + \sin \lambda \log(1 - 2r \cos \theta + r^2) \right\}.
\]

Since \( \{f_+(z)|t \in [-\pi, \pi]\} \) represent the extremal functions of Theorem 4, the distortion and rotation theorems follow from (31) through (35).

**Theorem 5.** If \( f(z) \in M^4(\beta) \), for fixed \( z \) (\(|z| = r < 1\)), \( T(r, \eta, \lambda, \beta) \leq \log |f(z)/z| \leq T(r, \eta_2, \lambda, \beta) \) where

\[
\eta_{1,2} = 2 \tan^{-1} \left\{ \frac{-\cot \lambda \mp (\csc^2 \lambda - r^2)^{1/2}}{1 + r} \right\}.
\]

**Proof.** It suffices to determine the bounds for \( \log |f_+(z)/z| \) where \( f_+(z) \) are the extremal functions for Theorem 4. Since \( \log |f_+(z)/z| = T(r, \eta, \lambda, \beta) \) is a real-valued function of \( \eta \), we may determine the maximum and minimum points by using elementary calculus. It follows that \( \partial T/\partial \eta = 0 \) for \( \eta_{1,2} \) given in (26). By examining \( \partial^2 T/\partial \eta^2 \), we find that \( \partial^2 T/\partial \eta^2 \) is positive for \( \eta_1 \) and negative for \( \eta_2 \). The result follows.

**Theorem 6.** If \( f(z) \in M^4(\beta) \) (\( z \) fixed, \(|z| = r < 1\)), then

\[ S(r, \eta_3, \lambda, \beta) \leq \arg f(z)/z \leq S(r, \eta_4, \lambda, \beta) \]

where

\[
\eta_{3,4} = 2 \tan^{-1} \left\{ \frac{\tan \lambda \mp (\sec^2 \lambda - r^2)^{1/2}}{1 + r} \right\}.
\]

**Proof.** This follows immediately by applying the same procedures as in the proof of Theorem 5 to \( \arg f_+(z)/z = S(r, \eta, \lambda, \beta) \). Here \( S(r, \eta, \lambda, \beta) \) is a real-valued function of \( \eta \) whose derivative is zero for \( \eta_{3,4} \)—given by (37). The second derivative of \( S \) is positive for \( \eta_3 \) and negative for \( \eta_4 \) from which the result follows.

**Remark.** For \( \beta = 0 \), Theorems 5 and 6 give us the known results for \( \lambda \)-spiral-like functions of order \( \beta \) [13].

**References**


Department of Mathematics, Clark University, Worcester, Massachusetts 01610

Current address: Department of Mathematics, University of California, Davis, California 95616