PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

ON A SUBCLASS OF SPIRAL-LIKE FUNCTIONS

E. M. SILVIA

ABSTRACT. Let $\alpha\geqq 0$, $0\leqq\beta<1$, $|\lambda|<\pi/2$ and suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is holomorphic in $U=\{z:|z|<1\}$. If

$$\operatorname{Re}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right)\right]>\beta\cos\lambda$$

for $z\in U$, then $f(z)$ is said to be α-λ-spiral-like of order β and we write $f(z)\in S_\alpha^\lambda(\beta)$. The author shows that for each $\alpha\geqq 0$, α-λ-spiral-like functions of order β are λ-spiral-like of order β. The following representation theorem is obtained: The function $f(z)\in S_\alpha^\lambda(\beta)$ $(\alpha>0, 0\leqq\beta<1, |\lambda|<\pi/2)$, if and only if there exists a function $F(\zeta)$ λ-spiral-like of order β such that

$$F(z)=\left[(e^{i\lambda}/\alpha)\int_0^z F(\zeta)^{e^{i\lambda}/\alpha}\zeta^{-1}\,d\zeta\right]^{\alpha e^{-i\lambda}}.$$

A distortion theorem for $\log|f(z)/z|$ and a rotation theorem for $\arg f(z)/z$ are also proved for functions $f(z)\in S_0^\lambda(\beta)$.

1. Let A denote the class of functions normalized by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in U $(|z|<1)$. For $0\leqq\beta<1$, we will let $S^*(\beta)$ represent the class of functions contained in A which are univalent and starlike of order β; i.e., $f(z)\in S^*(\beta)$ if $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is analytic and univalent satisfying $\operatorname{Re} zf'(z)/f(z)>\beta$ $(z\in U)$. Also, let P denote the class of analytic functions normalized by $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ such that $\operatorname{Re} p(z)>0$ $(z\in U)$.

A function $f(z)\in A$ is said to be spiral-like if there exists a λ $(|\lambda|<\pi/2)$ such that $\operatorname{Re} e^{i\lambda}zf'(z)/f(z)>0$ $(z\in U)$. L. Spaček defined the class of spiral-like functions in 1933 and showed that these functions are univalent **[15]**.

In 1967, R. Libera **[6]** extended this definition to functions spiral-like of order β. We say that $f(z)\in A$ is λ-spiral-like of order β $(0\leqq\beta<1, |\lambda|<\pi/2)$ if $\operatorname{Re} e^{i\lambda}zf'(z)/f(z)>\beta\cos\lambda$ $(z\in U)$.

A function $f(z)\in A$ satisfying $f(z)f'(z)\neq 0$ $(0<|z|<1)$ is said to be α-starlike of order β $(\alpha\geqq 0, 0\leqq\beta<1)$ if

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf''(z)}{f'(z)}+1\right)\right\}>\beta\qquad(z\in U).$$

Received by the editors March 1, 1973.
AMS (MOS) subject classifications (1970). Primary 30A36; Secondary 30A32.
Key words and phrases. Univalent, α-starlike, λ-spiral-like.

For $\beta=0$, we have the class of α-starlike functions (of order zero) which has been thoroughly investigated in **[7]**, **[8]**, **[9]**, **[10]**, and **[11]**. Some of these results have been extended to $0<\beta<1$ by the author **[13]**.

In this note, a class of functions which contains the classes of α-starlike functions of order β and λ-spiral-like functions of order β as special cases is defined; the functions in this new class will be shown to be λ-spiral-like. The author obtains an integral representation for the elements of this class in terms of λ-spiral-like functions of order β. Finally, a distortion and a rotation theorem for $f(z)/z$ whenever $f(z)$ is in this class is proved.

2. Just as the definition of λ-spiral-likeness of order β generalizes the definition of starlikeness of order β, we will generalize the definition of α-starlikeness of order β to α-λ-spiral-likeness of order β. In this section, we define the class of α-λ-spiral-like functions of order β—denoted $S_\alpha^\lambda(\beta)$—and show that each $f(z)\in S_\alpha^\lambda(\beta)$ is λ-spiral-like of order β.

Definition 1. Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in A$ and satisfy $f(z)f'(z)\neq 0$ in $0<|z|<1$. Set

$$K(\lambda, \alpha, f(z)) = (e^{i\lambda} - \alpha)zf'(z)/f(z) + \alpha(zf''(z)/f'(z) + 1). \tag{1}$$

Then $f(z)$ is said to be α-λ-spiral-like of order β if

$$\operatorname{Re} K(\lambda, \alpha, f(z)) > \beta \cos\lambda \qquad (z \in U) \tag{2}$$

where $\alpha\geqq 0$, $0\leqq\beta<1$, $|\lambda|<\pi/2$.

Remarks. (i) For $\alpha=0$, $S_0^\lambda(\beta)$ is the class of λ-spiral-like functions of order β.

(ii) For $\lambda=0=\beta$, we have $S_\alpha^0(0)$—the class of α-star-like functions (of order zero); while $S_\alpha^0(\beta)$ $(\alpha\geqq 0, 0\leqq\beta<1)$ is the class of α-starlike functions of order β.

In order to prove that α-λ-spiral-likeness of order β $(\alpha\geqq 0)$ implies λ-spiral-likeness of order β, we will need the following two lemmas: the first lemma is due to I. S. Jack **[4]** while the second is due to R. Libera **[6]**.

Lemma A. *Let $\omega(z)$ be regular in U with $\omega(0)=0$. If there exists a $\zeta\in U$ such that* $\operatorname{Max}_{|z|\leq|\zeta|}|\omega(z)|=|\omega(\zeta)|$, *then $\zeta\omega'(\zeta)=k\omega(\zeta)$ for some $k\geqq 1$.*

Lemma B. *The function $f(z)\in A$ is λ-spiral-like of order β $(0\leqq\beta<1$, $|\lambda|<\pi/2)$ if and only if there exists an $\omega(z)$ analytic satisfying $\omega(0)=0$, $|\omega(z)|<1$ such that*

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = \beta\cos\lambda + (1-\beta)\cos\lambda\left(\frac{1-\omega(z)}{1+\omega(z)}\right) + i\sin\lambda \quad (z\in U).$$

THEOREM 1. *If* $f(z) \in S_\alpha^\lambda(\beta)$ $(\alpha \geqq 0, 0 \leqq \beta < 1, |\lambda| < \pi/2)$ *then* $f(z)$ *is* λ*-spiral-like of order* β.

PROOF. Let

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = \beta\cos\lambda + (1-\beta)\cos\lambda\left(\frac{1-\omega(z)}{1+\omega(z)}\right) + i\sin\lambda. \tag{3}$$

Clearly, $\omega(0)=0$. In view of Lemma B, it suffices to show that $|\omega(z)|<1$. Simplifying (3), it follows that

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = \frac{e^{i\lambda}\{1 + (2\beta e^{-i\lambda}\cos\lambda - e^{-2i\lambda})\omega(z)\}}{1+\omega(z)}. \tag{4}$$

Differentiating (4) and using (1), we have

$$\begin{aligned} K(\lambda,\alpha,f(z)) &= \beta\cos\lambda + (1-\beta)\cos\lambda\left(\frac{1-\omega(z)}{1+\omega(z)}\right) + i\sin\lambda \\ &\quad + \alpha\frac{\{2\beta e^{-i\lambda}\cos\lambda - e^{-2i\lambda}\}z\omega'(z)}{1+(2\beta e^{-i\lambda}\cos\lambda - e^{-2i\lambda})\omega(z)} - \alpha\frac{z\omega'(z)}{1+\omega(z)}. \end{aligned} \tag{5}$$

Suppose that there exists a $\zeta \in U$ such that $\mathrm{Max}_{|z|\leqq|\zeta|}|\omega(z)| = |\omega(\zeta)| = 1$. Clearly $\omega(\zeta) \neq -1$. From Lemma A, there exists a $k \geqq 1$ such that $\zeta\omega'(\zeta) = k\omega(\zeta)$. For this ζ, we have

$$\mathrm{Re}(1-\omega(\zeta))/(1+\omega(\zeta)) = 0, \qquad \mathrm{Re}\,\zeta\omega(\zeta)/(1+\omega(\zeta)) = k/2. \tag{6}$$

Also, for

$$m = 2\beta e^{-i\lambda}\cos\lambda - e^{-2i\lambda}, \tag{7}$$

$$\begin{aligned} \mathrm{Re}\frac{m\zeta\omega'(\zeta)}{1+m\omega(\zeta)} &= \mathrm{Re}\frac{k(|m|^2 + m\omega(\zeta))}{1+|m|^2+2\,\mathrm{Re}\,m\omega(\zeta)} \\ &= \mathrm{Re}\frac{k(|m|^2 + \mathrm{Re}\,m\omega(\zeta))}{1+|m|^2+2\,\mathrm{Re}\,m\omega(\zeta)}. \end{aligned}$$

Hence,

$$\mathrm{Re}\left(\frac{m\zeta\omega'(\zeta)}{1+m\omega(\zeta)}\right) - \mathrm{Re}\left(\frac{\zeta\omega'(\zeta)}{1+\omega(\zeta)}\right) = \frac{k(|m|^2-1)}{2(1+2\,\mathrm{Re}\,m\omega(\zeta)+|m|^2)}. \tag{8}$$

Thus, from (6), (7) and (8), it follows that

$$\mathrm{Re}\,K(\lambda,\alpha,f(z)) = \beta\cos\lambda - \frac{2k\beta(1-\beta)\alpha\cos^2\lambda}{1+|m|^2+2\,\mathrm{Re}\,m\omega(\zeta)} < \beta\cos\lambda, \tag{9}$$

contradicting the assumption that $f(z) \in S_\alpha^\lambda(\beta)$. Therefore $|\omega(z)|<1$ in U and $f(z)$ is λ-spiral-like of order β.

COROLLARY. *If $f(z) \in S_\alpha^\lambda(\beta)$ then $f(z) \in S_\gamma^\lambda(\beta)$, $0 \leqq \gamma \leqq \alpha$.*

PROOF. By Theorem 1, $f(z) \in S_0^\lambda(\beta)$. Suppose there exists a γ, $0<\gamma<\alpha$, such that $f(z) \notin S_\gamma^\lambda(\beta)$. Then there is a $\zeta \in U$ for which

$$\text{(10)} \qquad \operatorname{Re}\left(\frac{\zeta f''(\zeta)}{f'(\zeta)} + 1 - \frac{\zeta f'(\zeta)}{f(\zeta)}\right) \leqq \frac{\beta \cos \lambda}{\gamma} - \frac{1}{\gamma} \operatorname{Re} \frac{\zeta f'(\zeta)}{f(\zeta)}.$$

However, for $f(z) \in S_\alpha^\lambda(\beta)$,

$$\text{(11)} \quad 0 < -\beta \cos \lambda + \operatorname{Re} e^{i\lambda} \frac{\zeta f'(\zeta)}{f(\zeta)} + \alpha \operatorname{Re}\left(\frac{\zeta f''(\zeta)}{f'(\zeta)} + 1 - \frac{\zeta f'(\zeta)}{f(\zeta)}\right).$$

Substituting (10) into (11), we obtain

$$0 < (1 - \alpha/\gamma)(\operatorname{Re} e^{i\lambda}\zeta f'(\zeta)/f(\zeta) - \beta \cos \lambda).$$

But $(1-\alpha/\gamma)<0$ implies $\operatorname{Re} e^{i\lambda} z f'(z)/f(z) < \beta \cos \lambda$, contradicting the assumption that $f(z) \in S_0^\lambda(\beta)$. Thus, $f(z) \in S_\gamma^\lambda(\beta)$.

3. In this section, the author obtains an important integral representation for the elements of $S_\alpha^\lambda(\beta)$. Throughout this section α, β, λ will represent constants such that $\alpha>0$, $0 \leqq \beta < 1$, $|\lambda|<\pi/2$.

DEFINITION 2. The function

$$f(z) = \left[(\gamma + i\mu) \int_0^z \sigma(t)^\gamma t^{-1+i\mu}\, dt\right]^{1/(\gamma+i\mu)}$$

where $\sigma(t) \in S^*(0)$, $\gamma>0$, μ real, $z \in U$ and the powers are meant as principal values, is called a Bazilevič function of type $\gamma+i\mu$. Denote the class of such functions by $B(\gamma+i\mu)$ [**2**].

Due to a result by Eenigenburg et al. [**3**], we know that each $f(z) \in B(\gamma+i\mu)$ is spiral-like. The functions that we will use in order to characterize the elements of $S_\alpha^\lambda(\beta)$ are those obtained when $\gamma=(\cos \lambda)/\alpha$ and $\mu=(\sin \lambda)/\alpha$.

DEFINITION 3. A function $f(z) \in A$ is said to be a Bazilevič function of type $e^{i\lambda}/\alpha$ and order β if

$$\text{(12)} \qquad f(z) = \left[\frac{e^{i\lambda}}{\alpha} \int_0^z \sigma(\zeta)^{(\cos \lambda)/\alpha} \zeta^{-1+((i \sin \lambda)/\alpha)}\, d\zeta\right]^{\alpha e^{-i\lambda}}$$

for some $\sigma(\zeta) \in S^*(\beta)$. Denote this by $f(z) \in B(e^{i\lambda}/\alpha, \beta)$.

Immediate from Definition 3 is

THEOREM 2. *If $f(z) \in B(e^{i\lambda}/\alpha, \beta)$ then $f(z) \in S_\alpha^\lambda(\beta)$.*

PROOF. For $f(z) \in B(e^{i\lambda}/\alpha, \beta)$, it follows from (12) that

$$f'(z) = \sigma(z)^{\cos\lambda/\alpha} z^{-1+((i\sin\lambda)/\alpha)} f(z)^{1-e^{i\lambda}/\alpha}. \tag{13}$$

Taking the logarithmic derivative of (13) we obtain an expression for $[zf''(z)/f'(z)]+1$. Substituting this into (1), we have

$$K(\lambda, \alpha, f(z)) = \cos\lambda z\sigma'(z)/\sigma(z) + i\sin\lambda. \tag{14}$$

Thus, $\operatorname{Re} K(\lambda, \alpha, f(z)) > \beta\cos\lambda$ or $f(z) \in S_\alpha^\lambda(\beta)$.

Using the following lemma due to Basgöze and Keogh **[1]**, a necessary and sufficient condition for $f(z)$ to be in $B(e^{i\lambda}/\alpha, \beta)$ is obtained.

LEMMA C. *A function $\sigma(\zeta) \in S^*(\beta)$ if and only if there exists a function $F(\zeta) \in S_0^\lambda(\beta)$ such that*

$$(\sigma(\zeta)/\zeta)^{\cos\lambda} = (F(\zeta)/\zeta)^{e^{i\lambda}} \qquad (\zeta \in U). \tag{15}$$

LEMMA 1. *A function $f(z) \in B(e^{i\lambda}/\alpha, \beta)$ if and only if there exists a function $F(\zeta) \in S_0^\lambda(\beta)$ such that*

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha}\int_0^z [F(\zeta)]^{e^{i\lambda}/\alpha}\zeta^{-1}\,d\zeta\right]^{\alpha e^{-i\lambda}} \tag{16}$$

where the powers are meant as principal values.

PROOF. From Definition 3, $f(z) \in B(e^{i\lambda}/\alpha, \beta)$ if and only if there exists a $\sigma(\zeta) \in S^*(\beta)$ satisfying (12). However, a necessary and sufficient condition for $\sigma(\zeta) \in S^*(\beta)$ is that there exists an $F(\zeta) \in S_0^\lambda(\beta)$ satisfying (15). Thus, for $f(z) \in B(e^{i\lambda}/\alpha, \beta)$, we may obtain

$$\begin{aligned} f(z) &= \left[\frac{e^{i\lambda}}{\alpha}\int_0^z \sigma(\zeta)^{(\cos\lambda)/\alpha}\zeta^{-1+i(\sin\lambda/\alpha)}\,d\zeta\right]^{\alpha e^{-i\lambda}} \\ &= \left[\frac{e^{i\lambda}}{\alpha}\int_0^z \left(\frac{\sigma(\zeta)}{\zeta}\right)^{(\cos\lambda)/\alpha}\zeta^{-1+(e^{i\lambda}/\alpha)}\,d\zeta\right]^{\alpha e^{-i\lambda}} \\ &= \left[\frac{e^{i\lambda}}{\alpha}\int_0^z [F(\zeta)]^{e^{i\lambda}/\alpha}\zeta^{-1}\,d\zeta\right]^{\alpha e^{-i\lambda}}, \end{aligned} \tag{17}$$

where $\sigma(\zeta)$ and $F(\zeta)$ are as above. Since each step in (17) is reversible, the result follows from this identity.

REMARK. From Lemma 1, a necessary and sufficient condition for $f(z) \in B(e^{i\lambda}/\alpha, \beta)$ is that

$$F(z) = f(z)[zf'(z)/f(z)]^{\alpha e^{-i\lambda}} \tag{18}$$

where $F(z) \in S_0^\lambda(\beta)$. Also, $B(e^{i\lambda}/\alpha, \beta) \subset S_\alpha^\lambda(\beta)$. In order to obtain the characterization for functions $f(z)\in S_\alpha^\lambda(\beta)$, we consider the converse problem. Given $F(\zeta)\in S_0^\lambda(\beta)$ and $\alpha>0$, when is the solution to the differential equation (18) with boundary condition $f(0)=0$, a function that is α-λ-spiral-like of order β? Since (18) may be rewritten as $[F(z)]^{e^{-i\lambda/\alpha}}/z= f'(z)f(z)^{-1+(e^{i\lambda}/\alpha)}$ we may perform the integration with boundary condition $f(0)=0$ to obtain

$$f(z)=\left[\frac{e^{i\lambda}}{\alpha}\int_0^z \frac{[F(\zeta)]^{e^{i\lambda}/\alpha}}{\zeta}\,d\zeta\right]^{\alpha e^{-i\lambda}}.$$

We will now show the proper definitions for which this formal solution is indeed an α-λ-spiral-like function of order β.

Lemma 2. *Let $f(z)\in S_\alpha^\lambda(\beta)$. For $0<\gamma\leqq\alpha$, choose the branch of $[zf'(z)/f(z)]^{\gamma e^{-i\lambda}}$ equal to 1 when $z=0$. Then the function*

$$F_\gamma(z)=f(z)[zf'(z)/f(z)]^{\gamma e^{-i\lambda}} \tag{19}$$

is λ-spiral-like of order β.

Proof. We have

$$e^{i\lambda}\frac{zF_\gamma'(z)}{F_\gamma(z)}=e^{i\lambda}\frac{zf'(z)}{f(z)}+\gamma\left(\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right)=K(\lambda,\gamma,f(z)).$$

But by the corollary to Theorem 1, we have that $f(z)\in S_\alpha^\lambda(\beta)$ implies $f(z)\in S_\gamma^\lambda(\beta)$ $(0\leqq\gamma\leqq\alpha)$. Therefore, $\operatorname{Re} e^{i\lambda}zF_\gamma'(z)/F_\gamma(z)=\operatorname{Re} K(\lambda,\gamma,f(z))>\beta\cos\lambda$ and $F_\gamma(z)\in S_0^\lambda(\beta)$.

Lemma 3. *If $F(z)=z+A_2z+\cdots\in S_0^\lambda(\beta)$ then $F(z)$ may be expressed as*

$$F(z)=f(z)[zf'(z)/f(z)]^{\alpha e^{-i\lambda}}, \tag{20}$$

where

$$f(z)=\left[\frac{e^{i\lambda}}{\alpha}\int_0^z [F(\zeta)]^{e^{i\lambda}/\alpha}\zeta^{-1}\,d\zeta\right]^{\alpha e^{-i\lambda}} \tag{21}$$

is an α-λ-spiral-like function of order β.

Proof. Let $h(z)=z^{-e^{i\lambda}/\alpha}\int_0^z [F(\zeta)]^{e^{i\lambda}/\alpha}\zeta^{-1}\,d\zeta$. We have

$$f(z)=z[(e^{i\lambda}/\alpha)h(z)]^{\alpha e^{-i\lambda}}$$

so that if $h(z)$ is independent of the path of integration it will follow that $f(z)$ is well defined.

Since $F(z)=z(1+A_2z+\cdots)\in S_0^\lambda(\beta)$, we have that $(1+A_2z+\cdots)$ is nonzero in U. Thus, we may write

$$(1 + A_2 z + \cdots)^{e^{i\lambda}/\alpha} = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{22}$$

for the power series expansion about $z=0$. From (22), it follows that

$$\int_0^z F(\zeta)^{e^{i\lambda}/\alpha} \zeta^{-1}\, d\zeta = \alpha e^{-i\lambda} z^{e^{i\lambda}/\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{c_n}{\alpha e^{i\lambda} n + 1} z^n + C\right). \tag{23}$$

To obtain a solution of (23) which is analytic and zero at the origin, take $C=0$. Thus, $h(z)=\alpha e^{-i\lambda}(1+\sum_{n=1}^{\infty} c_n z^n/(\alpha e^{i\lambda} n+1))$ is independent of the path of integration so that $f(z)$ given by (21) is well defined.

That $f(z)$ is α-λ-spiral-like of order β is a consequence of Theorem 2 and Lemma 1. Thus, the lemma is proved.

By combining the results of Theorem 2, Lemma 2 and Lemma 3, we have

THEOREM 3. *A necessary and sufficient condition for $f(z)$ to be in $S_\alpha^\lambda(\beta)$ is that $f(z)$ have the integral representation*

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha} \int_0^z [F(\zeta)]^{e^{i\lambda}/\alpha} \zeta^{-1}\, d\zeta\right]^{\alpha e^{-i\lambda}} \tag{24}$$

for some $F(\zeta) \in S_0^\lambda(\beta)$, where the powers are assumed to be principal values.

PROOF. If $f(z)$ is of the form (24), it follows immediately from Theorem 2 and Lemma 1 that $f(z) \in S_\alpha^\lambda(\beta)$. If $f(z) \in S_\alpha^\lambda(\beta)$, then—by Lemma 2 and Lemma 3—$f(z)$ can be written in the form (24).

Note that we now have $B(e^{i\lambda}/\alpha, \beta)=S_\alpha^\lambda(\beta)$ for $\alpha>0$, $0\leqq\beta<1$, $|\lambda|<\pi/2$.

4. We conclude this paper with a determination of a distortion theorem and a rotation theorem for $f(z)/z$ whenever $f(z) \in M_0^\lambda(\beta)=M^\lambda(\beta)$ $(0\leqq\beta<1$, $|\lambda|<\pi/2)$.

For $f(z) \in M^\lambda(\beta)$ $(0\leqq\beta<1$, $|\lambda|<\pi/2)$ there exists a $p(z) \in P$ such that

$$e^{i\lambda} z f'(z)/f(z) = (1-\beta)\cos\lambda\, p(z) + \beta\cos\lambda + i\sin\lambda. \tag{25}$$

From (25) it follows that

$$e^{i\lambda}(zf'(z)/f(z) - 1) = (1-\beta)\cos\lambda(p(z) - 1). \tag{26}$$

Throughout this section λ, β will denote constants satisfying $|\lambda|<\pi/2$, $0\leqq\beta<1$.

Using (26) we are able to obtain the convex hull of the image of $\log f(z)/z$ for fixed z $(|z|=r<1)$ when $f(z) \in M^\lambda(\beta)$.

THEOREM 4. *If $f(z) \in M^\lambda(\beta)$ then the set of all possible values of* $\log f(z)/z$ *(z fixed, $|z|=r<1$) lies in the image of $|z|\leqq r$ under the map*

$$\omega(z) = \log[(1-\varepsilon z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}], \qquad |\varepsilon| = 1. \tag{27}$$

PROOF. Dividing both sides of (26) by $z\neq 0$, integrating from 0 to z and simplifying, we have

$$\log\frac{f(z)}{z}=(1-\beta)e^{-i\lambda}\cos\lambda\int_0^z\frac{p(\zeta)-1}{\zeta}\,d\zeta. \tag{27}$$

Since $p(z)\in P$, Herglotz's theorem **[12]** may be applied to obtain

$$p(\zeta)=\int_{-\pi}^{\pi}\frac{1+\zeta e^{it}}{1-\zeta e^{it}}\,d\mu(t) \tag{28}$$

where $\mu(t)$ is nondecreasing in $[-\pi,\pi]$ and $\int_{-\pi}^{\pi}d\mu(t)=1$. From (28), it follows that

$$\frac{p(\zeta)-1}{\zeta}=\int_{-\pi}^{\pi}\frac{2e^{it}}{1-\zeta e^{it}}\,d\mu(t). \tag{29}$$

Substituting (29) into (27), we obtain

$$\log\frac{f(z)}{z}=-2(1-\beta)e^{-i\lambda}\cos\lambda\int_{-\pi}^{\pi}\log(1-e^{it}z)\,d\mu(t). \tag{30}$$

Let $q(z,t)=\log(1-e^{it}z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}$. Then $\operatorname{Re}\{1+zq''(z,t)/q'(z,t)\}=\operatorname{Re}[1/(1-ze^{it})]>\frac{1}{2}$. Thus, $q(z,t)$ is univalent in z and maps $|z|\leqq r<1$ onto a convex domain E, independent of t. From (30), we know that for fixed z ($|z|=r<1$) the points of $\log f(z)/z$ lie in the convex hull of E, denoted con E. However, since E is convex, $E=\text{con } E$ and the points of $\log f(z)/z$ (z fixed, $|z|=r<1$) lie in the convex image of $|z|\leqq r$ under the mapping $\omega(z)$ given by (27).

REMARKS. (i) For

$$\log f_t(z)/z=\log[(1-e^{it}z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}] \qquad (-\pi\leqq t<\pi),$$

we have

$$f_t(z)=z(1-e^{it}z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}.$$

These $f_t(z)$—for different t—are the extremal functions for Theorem 4.

(ii) We have

$$\log|f_t(z)/z|=\operatorname{Re}\log[(1-e^{it}z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}] \tag{31}$$

and

$$\arg f_t(z)/z=\operatorname{Im}\log[(1-e^{it}z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}]. \tag{32}$$

Also, for $z=re^{i\theta}$ ($0<r<1$, $0\leqq\theta<2\pi$) and $\eta=\theta+t$, we have

$$\log[(1-e^{it}z)^{-2(1-\beta)e^{-i\lambda}\cos\lambda}]=T(r,\eta,\lambda,\beta)+iS(r,\eta,\lambda,\beta) \tag{33}$$

where

$$T(r,\eta,\lambda,\beta) \tag{34}$$
$$= (1-\beta)\cos\lambda\left\{2\sin\lambda\arctan\frac{r\sin\eta}{1-r\cos\eta} - \cos\lambda\log(1-2r\cos\eta+r^2)\right\}$$

and

$$S(r,\eta,\lambda,\beta) \tag{35}$$
$$= (1-\beta)\cos\lambda\left\{2\cos\lambda\arctan\frac{r\sin\eta}{1-r\cos\eta} + \sin\lambda\log(1-2r\cos\eta+r^2)\right\}.$$

Since $\{f_t(z)|t \in [-\pi, \pi]\}$ represent the extremal functions of Theorem 4, the distortion and rotation theorems follow from (31) through (35).

THEOREM 5. *If $f(z) \in M^\lambda(\beta)$, for fixed z ($|z|=r<1$), $T(r, \eta_1, \lambda, \beta) \leqq \log|f(z)/z| \leqq T(r, \eta_2, \lambda, \beta)$ where*

$$\eta_{1,2} = 2\tan^{-1}\left\{\frac{-\cot\lambda \mp (\operatorname{cosec}^2\lambda - r^2)^{1/2}}{1+r}\right\}. \tag{36}$$

PROOF. It suffices to determine the bounds for $\log|f_t(z)/z|$ where $f_t(z)$ are the extremal functions for Theorem 4. Since $\log|f_t(z)/z| = T(r, \eta, \lambda, \beta)$ is a real-valued function of η, we may determine the maximum and minimum points by using elementary calculus. It follows that $\partial T/\partial\theta = 0$ for $\eta_{1,2}$ given in (26). By examining $\partial^2 T/\partial\theta^2$, we find that $\partial^2 T/\partial\theta^2$ is positive for η_1 and negative for η_2. The result follows.

THEOREM 6. *If $f(z) \in M^\lambda(\beta)$ (z fixed, $|z|=r<1$), then*

$$S(r, \eta_3, \lambda, \beta) \leqq \arg f(z)/z \leqq S(r, \eta_4, \lambda, \beta)$$

where

$$\eta_{3,4} = 2\tan^{-1}\left\{\frac{\tan\lambda \mp (\sec^2\lambda - r^2)^{1/2}}{1+r}\right\}. \tag{37}$$

PROOF. This follows immediately by applying the same procedures as in the proof of Theorem 5 to $\arg f_t(z)/z = S(r, \eta, \lambda, \beta)$. Here $S(r, \eta, \lambda, \beta)$ is a real-valued function of η whose derivative is zero for $\eta_{3,4}$—given by (37). The second derivative of S is positive for η_3 and negative for η_4 from which the result follows.

REMARK. For $\beta=0$, Theorems 5 and 6 give us the known results for λ-spiral-like functions of order β **[13]**.

REFERENCES

1. T. Basgöze and F. R. Keogh, *The Hardy class of a spiral-like function and its derivative*, Proc. Amer. Math. Soc. **26** (1970), 266–269. MR **41** #8680.

2. I. E. Bazilevič, *On a case of integrability in quadratures of the Loewner-Kufarev equation*, Mat. Sb. **37** **(79)** (1955), 471–476. (Russian) MR **17**, 356.

3. P. Eenigenburg, S. Miller, P. Mocanu and M. Reade, *On a class of Bazilevič functions*, Notices Amer. Math. Soc. **19** (1972), A706. Abstract #72T-B293.

4. I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. (2) **3** (1971), 469–474. MR **43** #7611.

5. R. Libera, *Univalent* α*-spiral functions*, Canad. J. Math. **19** (1967), 449–456. MR **35** #5599.

6. S. S. Miller, *Distortion properties of alpha-starlike functions*, Proc. Amer. Math. Soc. **38** (1973), 311–318.

7. S. S. Miller, P. T. Mocanu and M. O. Reade, *All* α*-convex functions are univalent and starlike*, Proc. Amer. Math. Soc. **37** (1973), 553–554.

8. ———, *Bazilevič functions and generalized convexity*, Rev. Roumaine Math. Pures Appl. (to appear).

9. P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) **11** (34), (1969), 127–133. MR **42** #7881.

10. P. T. Mocanu and M. O. Reade, *On generalized convexity of conformal mappings*, Rev. Roumaine Math. Pures Appl. **16** (1971), 1541–1544.

11. R. Nevanlinna, *Eindeutige analytische Funktionen*, Die Grundlehren der math. Wissenschaften, Band 46, Springer-Verlag, Berlin, 1936; English transl., Die Grundlehren der math. Wissenschaften, Band 162, Springer-Verlag, Berlin and New York, 1970. MR **43** #5003.

12. E. Silvia, *Classes related to* α*-starlike functions*, Ph.D. Dissertation, Clark University, Worcester, Mass., 1972.

13. R. Singh, *A note on spiral-like functions*, J. Indian Math. Soc. **33** (1969), 49–55. MR **41** #454.

14. L. Spaček, *Prispevek k teorii funkei prostych*, Časopis Pěst. Mat. **62** (1933), 12–19.

DEPARTMENT OF MATHEMATICS, CLARK UNIVERSITY, WORCESTER, MASSACHUSETTS 01610

Current address: Department of Mathematics, University of California, Davis, California 95616