ON THE BOUNDEDNESS OF $p$-INTEGRABLE AUTOMORPHIC FORMS

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Abstract. For a Fuchsian group, a criterion is obtained in order that every $p$-integrable automorphic form be bounded. This encompasses the known results for $p=1$. The condition implies an interesting inequality between the Bergman kernel and the Poincaré line element of a Riemann surface.

1. Introduction. Throughout, $\Gamma$ denotes a Fuchsian group acting on the unit disc $U$ of the complex plane. For any real number $q>1$, we choose and fix, once and for all, a system $\rho(q, T, z)$ ($z \in U$, $T \in \Gamma$) of factors of automorphy of dimension $-2q$ relative to $\Gamma$ (cf. [8]). Observe that, if $q$ is an integer, $\rho(q, T, z) = \chi(T)T'(z)^q$, where $\chi$ is a character of the group $\Gamma$.

Let $\lambda(z)dz \equiv |dz|/(1-|z|^2)$ be the Poincaré line element on $U$ and $d\omega(z) \equiv \lambda(z)dx
dy$ ($z=x+iy$, $x$, $y$ real) be the associated area element. For any $p$ with $1 \leq p \leq \infty$, let $A^p_{\omega}(\Gamma) \equiv A^p_{\omega}(\Gamma, \rho)$ be the Banach space of $p$-integrable holomorphic automorphic forms of dimension $-2q$ relative to $\Gamma$, i.e., the space of all $f$ that are holomorphic on $U$ and satisfy $f(Tz)\rho(q, T, z)=f(z)$, $\forall T \in \Gamma$, $z \in U$, and with finite $p$-norms $\|f\|_p$ defined by the formula

$$\|f\|_p = \iint_{\Omega} |\lambda^{-q}(z)f(z)|^p d\omega(z);$$

here, and, in the rest of the paper, $\Omega$ is a fundamental region for $\Gamma$ in $U'$ whose boundary has plane Lebesgue measure zero. It is a conjecture that, for arbitrary $\Gamma$ and $\rho$ (and $q>1$),

(1.1) \hspace{1cm} A^{p_1}_{\omega}(\Gamma) \subseteq A^{p_2}_{\omega}(\Gamma) \hspace{1cm} \text{if } 1 \leq p_1 < p_2 \leq \infty.

In this note, we obtain a condition for this inclusion to hold. For $p_1=1$ and $p_2=\infty$, the problem was considered by several authors (cf. [3], [4]).
and [6]-[9]). In §4 of this note, we point out an interesting inequality between the standard Bergman kernel of the Riemann surface $U/\Gamma$ and the Poincaré line element which is an immediate consequence of (1.1).

2. The criterion. For $z$ and $\zeta$ in $U$ and $q>1$, let

$$K(z, \zeta) \equiv K_{\rho}(z, \zeta) = (2q - 1)/\pi(1 - z\zeta)^{2q},$$

where $K(z, \zeta)$, for fixed $\zeta$, is analytic in $z$ and $K(0, \zeta)>0$. Then the Poincaré series

$$(2.1) \quad \alpha(z, \zeta) = \alpha_{q}(z, \zeta) = \sum_{T \in \Gamma} K(Tz, \zeta) \rho(q, T, z),$$

for fixed $\zeta$ in $U$, converges uniformly on compact subsets of $U$ and $\alpha(\cdot, \zeta)$ is a holomorphic automorphic form of dimension $-2q$ for $\Gamma$.

THEOREM. Let $q>1$. If $A_{\Gamma}^{p}(\gamma) \subset A_{\zeta}^{\infty}(\gamma)$ for some $p$ satisfying $1 \leq p < \infty$, then

$$(2.2) \quad M = M(\Gamma, p, q) = \sup_{z \in U} \lambda^{-2q}(z) \cdot \alpha(z, z) < \infty.$$

On the other hand, if (2.2) holds, then,

$$A_{\Gamma}^{p}(\gamma) \subset A_{\zeta}^{\infty}(\gamma) \quad \forall p, \quad 1 \leq p < \infty.$$

As an immediate consequence of the Theorem and Hölder's inequality, we obtain

COROLLARY 1. If (2.2) holds, then

$$A_{\Gamma}^{p_{1}}(\gamma) \subset A_{\zeta}^{p_{2}}(\gamma), \quad 1 \leq p_{1} < p_{2} \leq \infty.$$

Note. It is known (see below) that $\alpha(z, z) \geq 0$.

PROOF OF THE THEOREM. It is readily verified (cf. [8]) that

$$\|\alpha(\cdot, \zeta)\|_{1} \leq \int_{U} |\lambda^{-q}(z)K(z, \zeta)| \cdot d\omega(z)$$

$$(2.3) \quad = (2q - 1)/(q - 1) \cdot \lambda^{q}(\zeta) \equiv c_{q} \cdot \lambda^{q}(\zeta).$$

Also, since $K(\cdot, \zeta)$ is bounded on $U$, and $\sum_{T \in \Gamma} (1 - |Tz|^{2})^{q}$ is bounded on $U$ (cf. [1]), it follows that $\alpha(\cdot, \zeta) \in A_{\zeta}^{\infty}(\gamma)$. From Hölder's inequality, it then follows that $\alpha(\cdot, \zeta) \in A_{\zeta}^{p}(\gamma), \quad 1 \leq p \leq \infty$, and

$$(2.4) \quad \|\alpha(\cdot, \zeta)\|_{p} \leq \|\alpha(\cdot, \zeta)\|_{\infty}^{(p-1)/p} \|\alpha(\cdot, \zeta)\|_{1}^{1/p}.$$

Moreover, it is known (cf. Drasin [2, Equation (1.6)]) that, for $f$ in
\( A^p_q(\Gamma), \, 1 \leq p < \infty, \) the following reproducing formula holds:

\[
(2.5) \quad f(\zeta) = \int_{\Omega} f(z) \cdot \cos(\alpha(z, \zeta)) \cdot \lambda^{-2q}(z) \, d\omega(z).
\]

From this it follows that \( \alpha(z, \xi) = \cos(\alpha(\xi, z)) \) and \( \alpha(z, z) \geq 0. \) Schwarz's inequality, together with (2.5) implies that

\[
(2.6) \quad |\alpha(z, \zeta)|^2 \leq \alpha(z, z) \cdot \alpha(\zeta, \zeta).
\]

Let \( \beta(z) = \lambda^{-2q}(z) \cdot \alpha(z, z). \) It is readily verified that \( \beta \) is \( \Gamma \)-invariant, i.e., \( \beta \circ T = \beta, \, \forall T \in \Gamma, \) so that \( M = \sup_{z \in \Omega} \beta(z). \)

Suppose, now, that \( A^p_q(\Gamma) \subseteq A^\infty_q(\Gamma) \) for some \( p, \, 1 \leq p < \infty. \) Since convergence in either of the norms \( \| \|_p \) and \( \| \|_{\infty} \) implies uniform convergence on compact subsets of \( U, \) it follows that the inclusion map from \( A^p_q(\Gamma) \) to \( A^\infty_q(\Gamma) \) is a closed linear map of Banach spaces. But then the closed graph theorem implies that this is a bounded linear map, i.e., there exists a finite constant \( D = D(p, q, \Gamma, \rho) \) such that \( \| f \|_\infty \leq D \| f \|_p, \forall f \in A^p_q(\Gamma). \) In particular, \( \| \alpha(\cdot, \zeta) \|_{\infty} \leq D \cdot \| \alpha(\cdot, \zeta) \|_p. \) Combining this with (2.3) and (2.4) we conclude that

\[
\| \alpha(\cdot, \zeta) \|_{\infty} \leq D^p \cdot c_q \cdot \lambda^q(\zeta),
\]

i.e., \( \lambda^{-q}(z) \cdot |\alpha(z, \zeta)| \leq D^p c_q \cdot \lambda^q(\zeta), \, z, \, \zeta \in U. \) Setting \( z = \zeta \) in this, we obtain (2.2), thus establishing one half of the Theorem.

Conversely, suppose that (2.2) holds and let \( f \in A^p_q(\Gamma), \, 1 \leq p < \infty. \) From (2.5) and Hölder's inequality, we conclude that

\[
|f(\zeta)| \leq \| f \|_p \cdot \| \alpha(\cdot, \zeta) \|_{p'} \quad (1/p + 1/p' = 1)
\]

\[
\leq \| f \|_p \cdot \| \alpha(\cdot, \zeta) \|_{\infty}^{1-1/p'} \cdot [c_q \cdot \lambda(\zeta) q^{1/p'}]
\]

by (2.4) and (2.3). From (2.6) it now follows that,

\[
\lambda^{-q}(\zeta) \cdot |f(\zeta)| \leq \| f \|_p \cdot M^q \cdot c_q^{1/p'}.
\]

Since \( M \) is finite, this shows that \( \| f \|_{\infty} \) is finite, thus completing the proof of the Theorem.

3. The case of a large \( q. \) Let \( q \) be a real number \( > 1 \) and \( m \) a positive integer. Fix a system of factors of automorphy \( \rho(q, T, z) \) of dimension \( -2q. \) Then \( \rho^m(q, T, z) \) is a system of factors of automorphy for \( \Gamma \) of dimension \( -2qm \) and with these factors of automorphy, define \( A^p_{qm}(\Gamma) \) as above. We then have the following:

**Corollary 2.** Let \( A^p_{qm}(\Gamma) \subseteq A^\infty_{qm}(\Gamma) \) for some \( p, \, 1 \leq p < \infty; \) then \( A^p_q(\Gamma) \subseteq A^\infty_q(\Gamma). \)
Proof. It follows, as in the proof of the Theorem, that there exists a constant, \( B = B(q, m, \Gamma, \rho, p) \) such that \( \| f \|_\infty \leq B \cdot \| f \|_p \), \( \forall f \in A_{qm}^p \).

It is clear that \( \alpha^m(\cdot, \zeta) = \alpha^m_q(\cdot, \zeta) \in A_{qm}^p(\Gamma) \). Hence,

\[
\| \alpha^m(\cdot, \zeta) \|_\infty \leq B \cdot \| \alpha^m(\cdot, \zeta) \|_p,
\]

i.e.,

\[
\| \alpha(\cdot, \zeta) \|_\infty^m \leq B \| \alpha(\cdot, \zeta) \|_{mp}^m
\]

i.e.,

\[
\| \alpha(\cdot, \zeta) \|_\infty^{mp} \leq B^p \cdot \| \alpha(\cdot, \zeta) \|_{mp}^p.
\]

Combining this with (2.4) and (2.3), we conclude as above, that

\[
\| \alpha(\cdot, \zeta) \|_\infty \leq B^p \cdot c_\rho \cdot \lambda^q(\zeta).\]

This, as in the proof of the Theorem, implies (2.2) which, by the Theorem, is equivalent to the conclusion of Corollary 2.

4. Bergman kernel and the Poincaré metric. Consider factors of automorphy \( \rho(1, T, z) = T(z) \) and \( \rho(2, T, z) = T'(z)^2 \) of dimensions \(-2\) and \(-4\) respectively. With these factors of automorphy, consider the spaces \( A(\Gamma) \equiv A_1^2(\Gamma) \) and \( A_2^4(\Gamma) \) of automorphic forms. The space \( A(\Gamma) \) is a Hilbert space and can be identified with the space of square integrable Abelian differentials on the Riemann surface \( U/\Gamma \). It is readily verified that evaluation at a point is a bounded linear functional on \( A(\Gamma) \) and there exists, for each \( \zeta \in U \), a reproducing kernel \( k(\cdot, \zeta) \in A(\Gamma) \)—the Bergman kernel—such that

\[
f(\zeta) = \int_{\Omega} f(z) \cdot \operatorname{co}(k(z, \zeta)) \, dx \, dy \quad \forall f \in A(\Gamma).
\]

Corollary 3. If \( A_1^2(\Gamma) \subset A_2^2(\Gamma) \), then there exists a constant \( C = C(\Gamma) \) such that \( k(\zeta, \zeta) \leq C \lambda^q(\zeta), \forall \zeta \in U \).

Proof. Under the hypothesis, as in the proof of the Theorem, the linear injection \( A_1^2(\Gamma) \subset A_2^2(\Gamma) \) is continuous, and, hence, there exists a constant \( C \) such that \( \| f \|_\infty \leq C \| f \|_1, \forall f \in A_1^2(\Gamma) \). Clearly \( k^2(\cdot, \zeta) \in A_1^2(\Gamma) \). Hence,

\[
\| k^2(\cdot, \zeta) \|_\infty \leq C \| k^2(\cdot, \zeta) \|_1
\]

i.e.,

\[
C \cdot \| k(\cdot, \zeta) \|_2^2 = C k(\zeta, \zeta) \quad \text{by (4.1)}
\]

i.e.,

\[
(1 - |z|^2)^2 |k^2(z, \zeta)| \leq C k(\zeta, \zeta) \quad \forall z, \zeta \in U.
\]

Setting \( z = \zeta \), we have the conclusion.

5. Concluding remarks. (1) It is known (cf. [9]) that (2.2) holds if \( \Gamma \) is finitely generated. Lehner [7] has constructed a class of infinitely
generated groups for which $A^q_+(\Gamma) \subset A^\infty_+ (\Gamma)$, $q > 1$. For all such groups, then, the conclusion of Corollary 3 holds. (2) The results and proofs carry over if $\Gamma$ is replaced by a Kleinian group and $U$ by a $\Gamma$-invariant union of components of $\Gamma$ (cf. Kra [5]).

REFERENCES

7. ———, On the $A^q_+(\Gamma) \subset B^q_+ (\Gamma)$ conjecture in the theory of automorphic forms (to appear).

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