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ON THE BOUNDEDNESS OF p-INTEGRABLE AUTOMORPHIC FORMS

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ABSTRACT. For a Fuchsian group, a criterion is obtained in order that every p-integrable automorphic form be bounded. This encompasses the known results for $p=1$. The condition implies an interesting inequality between the Bergman kernel and the Poincaré line element of a Riemann surface.

1. **Introduction.** Throughout, Γ denotes a Fuchsian group acting on the unit disc U of the complex plane. For any real number $q>1$, we choose and fix, once and for all, a system $\rho(q, T, z)$ $(z \in U, T \in \Gamma)$ of factors of automorphy of dimension $-2q$ relative to Γ (cf. [8]). Observe that, if q is an integer, $\rho(q, T, z)=\chi(T)T'(z)^q$, where χ is a character of the group Γ.

Let $\lambda(z)|dz| \equiv |dz|/(1-|z|^2)$ be the Poincaré line element on U and $d\omega(z) \equiv \lambda^2(z)\, dx\, dy$ $(z=x+iy,\ x, y$ real) be the associated area element. For any p with $1 \leqq p \leqq \infty$, let $A_q^p(\Gamma) \equiv A_q^p(\Gamma, \rho)$ be the Banach space of p-integrable holomorphic automorphic forms of dimension $-2q$ relative to Γ, i.e., the space of all f that are holomorphic on U and satisfy $f(Tz)\rho(q, T, z)=f(z)$, $\forall T \in \Gamma$, $z \in U$, and with finite p-norms $\|f\|_p$ defined by the formula

$$\|f\|_p^p = \iint_\Omega |\lambda^{-q}(z) f(z)|^p \, d\omega(z);$$

here, and, in the rest of the paper, Ω is a fundamental region for Γ in U whose boundary has plane Lebesgue measure zero. It is a conjecture that, for arbitrary Γ and ρ (and $q>1$),

$$A_q^{p_1}(\Gamma) \subset A_q^{p_2}(\Gamma) \quad \text{if } 1 \leqq p_1 < p_2 \leqq \infty. \tag{1.1}$$

In this note, we obtain a condition for this inclusion to hold. For $p_1=1$ and $p_2=\infty$, the problem was considered by several authors (cf. [3], [4]

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and [6]–[9]). In §4 of this note, we point out an interesting inequality between the standard Bergman kernel of the Riemann surface U/Γ and the Poincaré line element which is an immediate consequence of (1.1).

2. **The criterion.** For z and ζ in U and $q>1$, let

$$K(z,\zeta) \equiv K_q(z,\zeta) = (2q-1)/\pi(1-z\bar{\zeta})^{2q},$$

where $K(z,\zeta)$, for fixed ζ, is analytic in z and $K(0,\zeta)>0$. Then the Poincaré series

$$\alpha(z,\zeta) = \alpha_q(z,\zeta) = \sum_{T\in\Gamma} K(Tz,\zeta)\rho(q,T,z), \tag{2.1}$$

for fixed ζ in U, converges uniformly on compact subsets of U and $\alpha(\cdot,\zeta)$ is a holomorphic automorphic form of dimension $-2q$ for Γ.

THEOREM. *Let $q>1$. If $A_q^p(\Gamma)\subset A_q^\infty(\Gamma)$ for some p satisfying $1\leqq p<\infty$, then*

$$M \equiv M(\Gamma,\rho,q) = \operatorname*{Sup}_{z\in U} \lambda^{-2q}(z)\cdot\alpha(z,z) < \infty. \tag{2.2}$$

On the other hand, if (2.2) holds, then,

$$A_q^p(\Gamma) \subset A_q^\infty(\Gamma) \quad \forall p, \qquad 1 \leqq p < \infty.$$

As an immediate consequence of the Theorem and Hölder's inequality, we obtain

COROLLARY 1. *If (2.2) holds, then*

$$A_q^{p_1}(\Gamma) \subset A_q^{p_2}(\Gamma), \qquad 1 \leqq p_1 < p_2 \leqq \infty.$$

Note. It is known (see below) that $\alpha(z,z)\geqq 0$.

PROOF OF THE THEOREM. It is readily verified (cf. [8]) that

$$\begin{aligned} \|\alpha(\cdot,\zeta)\|_1 &\leqq \iint_U |\lambda^{-q}(z)K(z,\zeta)|\cdot d\omega(z) \\ &= (2q-1)/(q-1)\cdot\lambda^q(\zeta) \equiv c_q\cdot\lambda^q(\zeta). \end{aligned} \tag{2.3}$$

Also, since $K(\cdot,\zeta)$ is bounded on U, and $\sum_{T\in\Gamma}(1-|Tz|^2)^q$ is bounded on U (cf. [1]), it follows that $\alpha(\cdot,\zeta)\in A_q^\infty(\Gamma)$. From Hölder's inequality, it then follows that $\alpha(\cdot,\zeta)\in A_q^p(\Gamma)$, $1\leqq p\leqq\infty$, and

$$\|\alpha(\cdot,\zeta)\|_p \leqq \|\alpha(\cdot,\zeta)\|_\infty^{(p-1)/p}\,\|\alpha(\cdot,\zeta)\|_1^{1/p}. \tag{2.4}$$

Moreover, it is known (cf. Drasin [2, Equation (1.6)]) that, for f in

$A_q^p(\Gamma)$, $1 \leqq p < \infty$, the following reproducing formula holds:

$$f(\zeta) = \iint_{\Omega} f(z) \cdot \mathrm{co}(\alpha(z, \zeta)) \cdot \lambda^{-2q}(z)\, d\omega(z). \tag{2.5}$$

From this it follows that $\alpha(z, \zeta) = \mathrm{co}(\alpha(\zeta, z))$ and $\alpha(z, z) \geqq 0$. Schwarz's inequality, together with (2.5) implies that

$$|\alpha(z, \zeta)|^2 \leqq \alpha(z, z) \cdot \alpha(\zeta, \zeta). \tag{2.6}$$

Let $\beta(z) = \lambda^{-2q}(z) \cdot \alpha(z, z)$. It is readily verified that β is Γ-invariant, i.e., $\beta \circ T = \beta$, $\forall T \in \Gamma$, so that $M = \mathrm{Sup}_{z \in \Omega}\, \beta(z)$.

Suppose, now, that $A_q^p(\Gamma) \subset A_q^\infty(\Gamma)$ for *some* p, $1 \leqq p < \infty$. Since convergence in either of the norms $\| \;\|_p$ and $\| \;\|_\infty$ implies uniform convergence on compact subsets of U, it follows that the inclusion map from $A_q^p(\Gamma)$ to $A_q^\infty(\Gamma)$ is a *closed* linear map of Banach spaces. But then the closed graph theorem implies that this is a bounded linear map, i.e., there exists a finite constant $D \equiv D(p, q, \Gamma, \rho)$ such that $\|f\|_\infty \leqq D \|f\|_p$, $\forall f \in A_q^p(\Gamma)$. In particular, $\|\alpha(\cdot, \zeta)\|_\infty \leqq D \cdot \|\alpha(\cdot, \zeta)\|_p$. Combining this with (2.3) and (2.4) we conclude that

$$\|\alpha(\cdot, \zeta)\|_\infty \leqq D^p \cdot c_q \cdot \lambda^q(\zeta),$$

i.e., $\lambda^{-q}(z) \cdot |\alpha(z, \zeta)| \leqq D^p c_q \cdot \lambda^q(\zeta)$, $z, \zeta \in U$. Setting $z = \zeta$ in this, we obtain (2.2), thus establishing one half of the Theorem.

Conversely, suppose that (2.2) holds and let $f \in A_q^p(\Gamma)$, $1 \leqq p < \infty$. From (2.5) and Hölder's inequality, we conclude that

$$\begin{aligned} |f(\zeta)| &\leqq \|f\|_p \cdot \|\alpha(\cdot, \zeta)\|_{p'} \qquad (1/p + 1/p' = 1) \\ &\leqq \|f\|_p \cdot \|\alpha(\cdot, \zeta)\|_\infty^{1-1/p'} \cdot [c_q \cdot \lambda(\zeta)^q]^{1/p'} \end{aligned}$$

by (2.4) and (2.3). From (2.6) it now follows that,

$$\lambda^{-q}(\zeta)\, |f(\zeta)| \leqq \|f\|_{p'}\, M^{1-1/p'} \cdot c_q^{1/p'}.$$

Since M is finite, this shows that $\|f\|_\infty$ is finite, thus completing the proof of the Theorem.

3. **The case of a large** q. Let q be a real number >1 and m a positive integer. Fix a system of factors of automorphy $\rho(q, T, z)$ of dimension $-2q$. Then $\rho^m(q, T, z)$ is a system of factors of automorphy for Γ of dimension $-2qm$ and with these factors of automorphy, define $A_{qm}^p(\Gamma)$ as above. We then have the following:

Corollary 2. *Let* $A_{qm}^p(\Gamma) \subset A_{qm}^\infty(\Gamma)$ *for some* p, $1 \leqq p < \infty$; *then* $A_q^p(\Gamma) \subset A_q^\infty(\Gamma)$.

PROOF. It follows, as in the proof of the Theorem, that there exists a constant, $B \equiv B(q, m, \Gamma, \rho, p)$ such that $\|f\|_\infty \leqq B \cdot \|f\|_p$, $\forall f \in A^p_{qm}$. It is clear that $\alpha^m(\cdot, \zeta) \equiv \alpha^m_q(\cdot, \zeta) \in A^p_{qm}(\Gamma)$. Hence,

$$\|\alpha^m(\cdot, \zeta)\|_\infty \leqq B \cdot \|\alpha^m(\cdot, \zeta)\|_p,$$

i.e.,

$$\|\alpha(\cdot, \zeta)\|^m_\infty \leqq B\, \|\alpha(\cdot, \zeta)\|^m_{mp}$$

i.e.,

$$\|\alpha(\cdot, \zeta)\|^{mp}_\infty \leqq B^p \cdot \|\alpha(\cdot, \zeta)\|^{mp}_{mp}.$$

Combining this with (2.4) and (2.3), we conclude as above, that

$$\|\alpha(\cdot, \zeta)\|_\infty \leqq B^p \cdot c_q \cdot \lambda^q(\zeta).$$

This, as in the proof of the Theorem, implies (2.2) which, by the Theorem, is equivalent to the conclusion of Corollary 2.

4. **Bergman kernel and the Poincaré metric.** Consider factors of automorphy $\rho(1, T, z) = T'(z)$ and $\rho(2, T, z) = T'(z)^2$ of dimensions -2 and -4 respectively. With these factors of automorphy, consider the spaces $A(\Gamma) \equiv A^2_1(\Gamma)$ and $A^1_2(\Gamma)$ of automorphic forms. The space $A(\Gamma)$ is a Hilbert space and can be identified with the space of square integrable Abelian differentials on the Riemann surface U/Γ. It is readily verified that evaluation at a point is a bounded linear functional on $A(\Gamma)$ and there exists, for each $\zeta \in U$, a reproducing kernel $k(\cdot, \zeta) \in A(\Gamma)$—the Bergman kernel—such that

$$f(\zeta) = \iint_\Omega f(z) \cdot \mathrm{co}(k(z, \zeta))\, dx\, dy \quad \forall f \in A(\Gamma). \tag{4.1}$$

COROLLARY 3. *If $A^1_2(\Gamma) \subset A^\infty_2(\Gamma)$, then there exists a constant $C = C(\Gamma)$ such that $k(\zeta, \zeta) \leqq C\lambda^2(\zeta)$, $\forall \zeta \in U$.*

PROOF. Under the hypothesis, as in the proof of the Theorem, the linear injection $A^1_2(\Gamma) \subset A^\infty_2(\Gamma)$ is continuous, and, hence, there exists a constant C such that $\|f\|_\infty \leqq C\|f\|_1$, $\forall f \in A^1_2(\Gamma)$. Clearly $k^2(\cdot, \zeta) \in A^1_2(\Gamma)$. Hence,

$$\begin{aligned}\|k^2(\cdot, \zeta)\|_\infty &\leqq C\, \|k^2(\cdot, \zeta)\|_1 \\ &= C \cdot \|k(\cdot, \zeta)\|^2_2 = Ck(\zeta, \zeta) \quad \text{by (4.1)}\end{aligned}$$

i.e.,

$$(1 - |z|^2)^2\, |k^2(z, \zeta)| \leqq Ck(\zeta, \zeta) \quad \forall z, \zeta \in U.$$

Setting $z = \zeta$, we have the conclusion.

5. **Concluding remarks.** (1) It is known (cf. [9]) that (2.2) holds if Γ is finitely generated. Lehner [7] has constructed a class of infinitely

generated groups for which $A_q^1(\Gamma) \subset A_q^\infty(\Gamma)$, $q > 1$. For all such groups, then, the conclusion of Corollary 3 holds. (2) The results and proofs carry over if Γ is replaced by a Kleinian group and U by a Γ-invariant union of components of Γ (cf. Kra [**5**]).

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