ON THE BOUNDEDNESS OF p-INTEGRABLE
AUTOMORPHIC FORMS

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ABSTRACT. For a Fuchsian group, a criterion is obtained in order that every p-integrable automorphic form be bounded. This encompasses the known results for $p=1$. The condition implies an interesting inequality between the Bergman kernel and the Poincaré line element of a Riemann surface.

1. **Introduction.** Throughout, Γ denotes a Fuchsian group acting on the unit disc U of the complex plane. For any real number $q>1$, we choose and fix, once and for all, a system $\rho(q, T, z)$ ($z \in U$, $T \in \Gamma$) of factors of automorphy of dimension $-2q$ relative to Γ (cf. [8]). Observe that, if q is an integer, $\rho(q, T, z)=\chi(T)T'(z)^q$, where χ is a character of the group Γ.

Let $\lambda(z)|dz| \equiv |dz|/(1-|z|^2)$ be the Poincaré line element on U and $d\omega(z) \equiv \lambda^2(z)\,dx\,dy$ ($z=x+iy$, x, y real) be the associated area element. For any p with $1 \leqq p \leqq \infty$, let $A_q^p(\Gamma) \equiv A_q^p(\Gamma, \rho)$ be the Banach space of p-integrable holomorphic automorphic forms of dimension $-2q$ relative to Γ, i.e., the space of all f that are holomorphic on U and satisfy $f(Tz)\rho(q, T, z)=f(z)$, $\forall T \in \Gamma$, $z \in U$, and with finite p-norms $\|f\|_p$ defined by the formula

$$\|f\|_p^p = \iint\limits_{\Omega} |\lambda^{-q}(z)f(z)|^p \, d\omega(z);$$

here, and, in the rest of the paper, Ω is a fundamental region for Γ in U' whose boundary has plane Lebesgue measure zero. It is a conjecture that, for arbitrary Γ and ρ (and $q>1$),

(1.1) $\qquad A_q^{p_1}(\Gamma) \subset A_q^{p_2}(\Gamma)$ if $1 \leqq p_1 < p_2 \leqq \infty$.

In this note, we obtain a condition for this inclusion to hold. For $p_1=1$ and $p_2=\infty$, the problem was considered by several authors (cf. [3], [4]

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and [6]–[9]). In §4 of this note, we point out an interesting inequality between the standard Bergman kernel of the Riemann surface U/Γ and the Poincaré line element which is an immediate consequence of (1.1).

2. **The criterion.** For z and ζ in U and $q>1$, let

$$K(z, \zeta) \equiv K_q(z, \zeta) = (2q - 1)/\pi(1 - z\bar{\zeta})^{2q},$$

where $K(z, \zeta)$, for fixed ζ, is analytic in z and $K(0, \zeta)>0$. Then the Poincaré series

(2.1) $$\alpha(z, \zeta) = \alpha_q(z, \zeta) = \sum_{T\epsilon\Gamma} K(Tz, \zeta)\rho(q, T, z),$$

for fixed ζ in U, converges uniformly on compact subsets of U and $\alpha(\cdot, \zeta)$ is a holomorphic automorphic form of dimension $-2q$ for Γ.

THEOREM. *Let $q>1$. If $A_q^p(\Gamma)\subset A_q^\infty(\Gamma)$ for **some** p satisfying $1\leqq p<\infty$, then*

(2.2) $$M \equiv M(\Gamma, \rho, q) = \operatorname*{Sup}_{z\epsilon U} \lambda^{-2q}(z) \cdot \alpha(z, z) < \infty.$$

On the other hand, if (2.2) holds, then,

$$A_q^p(\Gamma) \subset A_q^\infty(\Gamma) \quad \forall p, \qquad 1 \leqq p < \infty.$$

As an immediate consequence of the Theorem and Hölder's inequality, we obtain

COROLLARY 1. *If (2.2) holds, then*

$$A_q^{p_1}(\Gamma) \subset A_q^{p_2}(\Gamma), \qquad 1 \leqq p_1 < p_2 \leqq \infty.$$

Note. It is known (see below) that $\alpha(z, z)\geqq 0$.

PROOF OF THE THEOREM. It is readily verified (cf. [8]) that

(2.3)
$$\|\alpha(\cdot, \zeta)\|_1 \leqq \iint_U |\lambda^{-q}(z)K(z, \zeta)| \cdot d\omega(z)$$
$$= (2q - 1)/(q - 1) \cdot \lambda^q(\zeta) \equiv c_q \cdot \lambda^q(\zeta).$$

Also, since $K(\cdot, \zeta)$ is bounded on U, and $\sum_{T\epsilon\Gamma} (1-|Tz|^2)^q$ is bounded on U (cf. [1]), it follows that $\alpha(\cdot, \zeta) \in A_q^\infty(\Gamma)$. From Hölder's inequality, it then follows that $\alpha(\cdot, \zeta) \in A_q^p(\Gamma), 1\leqq p\leqq\infty$, and

(2.4) $$\|\alpha(\cdot, \zeta)\|_p \leqq \|\alpha(\cdot, \zeta)\|_\infty^{(p-1)/p} \|\alpha(\cdot, \zeta)\|_1^{1/p}.$$

Moreover, it is known (cf. Drasin [2, Equation (1.6)]) that, for f in

$A_q^p(\Gamma)$, $1 \leqq p < \infty$, the following reproducing formula holds:

$$(2.5) \qquad f(\zeta) = \iint_\Omega f(z) \cdot \mathrm{co}(\alpha(z, \zeta)) \cdot \lambda^{-2q}(z) \, d\omega(z).$$

From this it follows that $\alpha(z, \zeta) = \mathrm{co}(\alpha(\zeta, z))$ and $\alpha(z, z) \geqq 0$. Schwarz's inequality, together with (2.5) implies that

$$(2.6) \qquad |\alpha(z, \zeta)|^2 \leqq \alpha(z, z) \cdot \alpha(\zeta, \zeta).$$

Let $\beta(z) = \lambda^{-2q}(z) \cdot \alpha(z, z)$. It is readily verified that β is Γ-invariant, i.e., $\beta \circ T = \beta$, $\forall T \in \Gamma$, so that $M = \mathrm{Sup}_{z \in \Omega}\, \beta(z)$.

Suppose, now, that $A_q^p(\Gamma) \subset A_q^\infty(\Gamma)$ for *some* p, $1 \leqq p < \infty$. Since convergence in either of the norms $\| \ \|_p$ and $\| \ \|_\infty$ implies uniform convergence on compact subsets of U, it follows that the inclusion map from $A_q^p(\Gamma)$ to $A_q^\infty(\Gamma)$ is a *closed* linear map of Banach spaces. But then the closed graph theorem implies that this is a bounded linear map, i.e., there exists a finite constant $D \equiv D(p, q, \Gamma, \rho)$ such that $\|f\|_\infty \leqq D\|f\|_p$, $\forall f \in A_q^p(\Gamma)$. In particular, $\|\alpha(\cdot, \zeta)\|_\infty \leqq D \cdot \|\alpha(\cdot, \zeta)\|_p$. Combining this with (2.3) and (2.4) we conclude that

$$\|\alpha(\cdot, \zeta)\|_\infty \leqq D^p \cdot c_q \cdot \lambda^q(\zeta),$$

i.e., $\lambda^{-q}(z) \cdot |\alpha(z, \zeta)| \leqq D^p c_q \cdot \lambda^q(\zeta)$, $z, \zeta \in U$. Setting $z = \zeta$ in this, we obtain (2.2), thus establishing one half of the Theorem.

Conversely, suppose that (2.2) holds and let $f \in A_q^p(\Gamma)$, $1 \leqq p < \infty$. From (2.5) and Hölder's inequality, we conclude that

$$|f(\zeta)| \leqq \|f\|_p \cdot \|\alpha(\cdot, \zeta)\|_{p'} \qquad (1/p + 1/p' = 1)$$
$$\leqq \|f\|_p \cdot \|\alpha(\cdot, \zeta)\|_\infty^{1 - 1/p'} \cdot [c_q \cdot \lambda(\zeta)^q]^{1/p'}$$

by (2.4) and (2.3). From (2.6) it now follows that,

$$\lambda^{-q}(\zeta)\, |f(\zeta)| \leqq \|f\|_{p'} \cdot M^{1 - 1/p'} \cdot c_q^{1/p'}.$$

Since M is finite, this shows that $\|f\|_\infty$ is finite, thus completing the proof of the Theorem.

3. **The case of a large q.** Let q be a real number > 1 and m a positive integer. Fix a system of factors of automorphy $\rho(q, T, z)$ of dimension $-2q$. Then $\rho^m(q, T, z)$ is a system of factors of automorphy for Γ of dimension $-2qm$ and with these factors of automorphy, define $A_{qm}^p(\Gamma)$ as above. We then have the following:

COROLLARY 2. *Let* $A_{qm}^p(\Gamma) \subset A_{qm}^\infty(\Gamma)$ *for some* p, $1 \leqq p < \infty$; *then* $A_q^p(\Gamma) \subset A_q^\infty(\Gamma)$.

PROOF. It follows, as in the proof of the Theorem, that there exists a constant, $B \equiv B(q, m, \Gamma, \rho, p)$ such that $\|f\|_\infty \leq B \cdot \|f\|_p$, $\forall f \in A_{qm}^p$. It is clear that $\alpha^m(\cdot, \zeta) \equiv \alpha_q^m(\cdot, \zeta) \in A_{qm}^p(\Gamma)$. Hence,

$$\|\alpha^m(\cdot, \zeta)\|_\infty \leq B \cdot \|\alpha^m(\cdot, \zeta)\|_p,$$

i.e.,

$$\|\alpha(\cdot, \zeta)\|_\infty^m \leq B \|\alpha(\cdot, \zeta)\|_{mp}^m$$

i.e.,

$$\|\alpha(\cdot, \zeta)\|_\infty^{mp} \leq B^p \cdot \|\alpha(\cdot, \zeta)\|_{mp}^{mp}.$$

Combining this with (2.4) and (2.3), we conclude as above, that

$$\|\alpha(\cdot, \zeta)\|_\infty \leq B^p \cdot c_q \cdot \lambda^q(\zeta).$$

This, as in the proof of the Theorem, implies (2.2) which, by the Theorem, is equivalent to the conclusion of Corollary 2.

4. Bergman kernel and the Poincaré metric. Consider factors of automorphy $\rho(1, T, z) = T'(z)$ and $\rho(2, T, z) = T'(z)^2$ of dimensions -2 and -4 respectively. With these factors of automorphy, consider the spaces $A(\Gamma) \equiv A_1^2(\Gamma)$ and $A_2^1(\Gamma)$ of automorphic forms. The space $A(\Gamma)$ is a Hilbert space and can be identified with the space of square integrable Abelian differentials on the Riemann surface U/Γ. It is readily verified that evaluation at a point is a bounded linear functional on $A(\Gamma)$ and there exists, for each $\zeta \in U$, a reproducing kernel $k(\cdot, \zeta) \in A(\Gamma)$—the Bergman kernel—such that

$$(4.1) \qquad f(\zeta) = \iint_\Omega f(z) \cdot \mathrm{co}(k(z, \zeta)) \, dx \, dy \quad \forall f \in A(\Gamma).$$

COROLLARY 3. *If $A_2^1(\Gamma) \subset A_2^\infty(\Gamma)$, then there exists a constant $C = C(\Gamma)$ such that $k(\zeta, \zeta) \leq C\lambda^2(\zeta)$, $\forall \zeta \in U$.*

PROOF. Under the hypothesis, as in the proof of the Theorem, the linear injection $A_2^1(\Gamma) \subset A_2^\infty(\Gamma)$ is continuous, and, hence, there exists a constant C such that $\|f\|_\infty \leq C\|f\|_1$, $\forall f \in A_2^1(\Gamma)$. Clearly $k^2(\cdot, \zeta) \in A_2^1(\Gamma)$. Hence,

$$\|k^2(\cdot, \zeta)\|_\infty \leq C \|k^2(\cdot, \zeta)\|_1$$
$$= C \cdot \|k(\cdot, \zeta)\|_2^2 = Ck(\zeta, \zeta) \quad \text{by (4.1)}$$

i.e.,

$$(1 - |z|^2)^2 |k^2(z, \zeta)| \leq Ck(\zeta, \zeta) \quad \forall z, \zeta \in U.$$

Setting $z = \zeta$, we have the conclusion.

5. Concluding remarks. (1) It is known (cf. [9]) that (2.2) holds if Γ is finitely generated. Lehner [7] has constructed a class of infinitely

generated groups for which $A_q^1(\Gamma) \subset A_q^\infty(\Gamma)$, $q > 1$. For all such groups, then, the conclusion of Corollary 3 holds. (2) The results and proofs carry over if Γ is replaced by a Kleinian group and U by a Γ-invariant union of components of Γ (cf. Kra [5]).

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